A cognitive view of relevant implication

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Abstract. Relevant logics provide an alternative to classical implication that is capable of accounting for the relationship between the antecedent and the consequence of a valid implication. Relevant implication is usually explained in terms of information required to assess a proposition. By doing so, relevant implication introduces a number of cognitively relevant aspects in the definition of logical operators. In this paper, we aim to take a closer look at the cognitive feature of relevant implication. For this purpose, we develop a cognitively-oriented interpretation of the semantics of relevant logics. In particular, we provide an interpretation of Routley-Meyer semantics in terms of conceptual spaces and we show that it meets the constraints of the algebraic semantics of relevant logic.

1 Introduction

Paradoxes of classical material implication often show a mismatch between our intuitions concerning valid patterns of reasoning and the formalization of implication provided by classical logic. Debates on the nature of implication can be traced back to the very origin of modern logic, involving for instance Brentano, Husserl, and Frege. Turning to contemporary developments of mathematical logic, the problem of the logical properties of implication has been approached by providing systems that aims to mend classical logic from inference patterns that are not motivated on the basis of a specific view of reasoning.

Since in any logical system, the implication has the important role of encoding the properties of logical inference, by rejecting the properties of classical implication, one is often lead to rejecting classical logic. For instance, *intuitionistic logic* criticizes the non-constructive nature of classical implication. For that reason, intuitionists designed an alternative logic that rejects inference by contradiction and the law of the excluded middle. Moreover, *relevant logic* criticizes the lack of connection between the premises and the conclusion of a logical inference made explicit by some valid formula of classical logic, e.g., $A \to (B \to A)$—once A holds, one can infer that any B entails A—or $(A \to B) \vee (B \to A)$—every pair of propositions can be connected by means of an implication. By keeping track of the antecedent-consequent connection, relevant logic prevents these paradoxes.

Furthermore, classical implication does not model any sort of relationship between the knowing subject and the matter of the proposition. The truth-conditional definition of the classical implication $A \to B$ is given in terms of those states of affairs such that either the state of affairs corresponding to A does not hold or the state of affairs corresponding to B holds. Prosaically, $A \to B$ is true whenever A is false or B is true. The relationship between the antecedent and the consequent of a classical implication can be understood only in terms of mere co-occurrence between the states of affairs of the corresponding propositions. The knowing subject is construed as a spectator of an independent reality that displays itself. A number of approaches to non-classical logics can be categorized as proposals to make logical implication sensitive to *cognitively relevant* aspects. For instance, intuitionistic logic models the abstract concept of a knowing subject and intuitionistic semantics is better understood in terms of *proof-conditions* instead of truth-conditions, where a proof is intended to model the activity of a knowing subject [20] with respect to propositions. A significant number of non-classical logics are motivated by the idea of taking into account the activity of the knowing subject, e.g., just to mention a few, *justification logics* [8], *proof-theoretical semantics* [21], and number of *relevant* and *substructural logics* [13, 5]. Each of this approaches stresses that the information required to asses the status of a proposition is an essential part of the meaning of the proposition.

We place our analysis within the tradition of relevant logics [13, 2], a family of logics that have been traditionally interpreted as logics of information [12, 1, 13]. In particular, the analysis of relevant implication aims to investigate the connection between the information contained in the antecedent and the information contained in the consequence. Although relevant logics are effective in preventing paradoxes of material implication, a drawback is that their algebraic semantics has been criticized on the ground that it lacks any strong intuitive motivation [5]. To cope with that, a number of approaches to relevant logics provided an intuitive reading of the semantics. From the point of view of cognition, the most interesting approach is due to Mares [13] who interprets deduction in relevant logics in terms of *situated inference*. Intuitively, a situation contains information that is relevant to make a proposition hold, thus situations are truth-makers of propositions. In this paper, we provide a version of the semantics of relevant logic based on a notion of situation defined in terms of the theory of *conceptual spaces* [9], a theory on how we conceptualize the reality and how we reason on this conceptualization. Our aim is to motivate the idea of situated inference provided by Mares by means of the rich theory of cognition formalized by means of conceptual spaces. The exhibition of a concrete instance of the semantics of relevant logics based on a well developed model of cognition has a double impact: (*i*) it provides a clean cognitive interpretation of relevant logics; and (*ii*) it shows that relevant logics capture cognitively important aspects of inferences.

The paper is organized as follows. Sections 2 and 3 introduce the background on relevant logic and conceptual spaces. Section 4 informally describes the interpretation of the semantics of relevant logic in terms of conceptual spaces, while Section 5 provides the formal construction. Section 6 concludes the paper.

2 Relevant logic

We introduce a minimal background on the relevant logic **R** [2, 13, 7]. We confine ourself to the implicative fragment of **R** that, by slightly abusing the notation,

1. $A \to A$
2. $(A \to B) \to ((B \to C) \to (A \to C))$
3. $A \to ((A \to B) \to B)$
4. $(A \to (A \to B)) \to (A \to B)$

Table 1. Axioms for **R**

we still label by **R**. Let *Atom* be a set of propositional atoms and $p \in Atom$, the language of **R** is inductively defined by:

$$L_{\mathbf{R}} := p \mid A \to A$$

The axioms for **R** are presented in Table 1 while its Hilbert system is introduced as usual through the notion of derivation $\vdash_{\mathbf{R}}$. The base case states that $\vdash_{\mathbf{R}} \phi$, where ϕ is an axiom in Table 1. The rule of *modus ponens* is then added: if $\vdash_{\mathbf{R}} A$, $\vdash_{\mathbf{R}} A \to B$, then $\vdash_{\mathbf{R}} B$. By reasoning in **R**, a number of paradoxes of classical implication are blocked. For instance, the monotonicity of the entailment $A \to (B \to A)$, which is an axiom in classical logic. Its meaning is: if A holds, then every B entails A, regardless the relevance of B for assessing A. Accordingly, in relevant logics that axiom is not valid. Moreover, in case we also assume a disjunction in our language, $(A \to B) \vee (B \to A)$ is not a theorem of **R**.

2.1 Routley-Meyer Semantics

We present the model of substructural logic in terms of ternary relations, that is due to Routley and Meyer [15, 18]. Ternary relations can be viewed as a generalization of (relational) Kripke semantics for intuitionistic and modal logics. Let S be a set of points and $R \subseteq S^3$. Moreover, let $1 \in S$ be a designated element. We define the following notations:

- $R^2(xy)zw$ iff there is an $u \in S$ such that $Rxyu$ and $Ruzw$;
- $x \leq y$ iff $R1xy$.

Definition 1 (Substructural frame)). *A substructural frame* $\mathcal{S} = (S, 1, R)$ *is a set* S*, with* $1 \in S$*, equipped with a ternary relation* R *such that:*

A1. $x \leq x$ $(R1xx)$
A2. $Rxxx$
A3. if $R^2(xy)zw$*, then* $R^2(xz)yw$
(*if there is* u *s.t.* $Rxyu$ *and* $Ruzw$*, then there is* v *s.t.* $Rxzv$ *and* $Rvzw$)
A4. if $Rxyz$*, then* $Ryxz$
A5. if $Rxyz$ *and* $x \leq w$*, then* $Rwyz$

A valuation in a substructural frame is defined by $v : Atom \to \mathcal{P}(S)$. The valuation is required to satisfy the following *heredity condition*: for every $p \in Atom$, if $x \in v(p)$ and $x \leq y$, then $y \in v(p)$. The valuation extends to any formula of R, by the semantics of implication:

– $s \models A \rightarrow B$ iff for all r,t such that $Rsrt$, if $r \models A$, then $t \models B$.

Heredity has to extend to complex formulas, and it is easy to check that it is the case. The concept of truth in a model is defined by evaluating propositions at the particular designed state 1.

Definition 2 (Substructural model). *A substructural model $(\mathcal{S}, v)$ is a substructural frame $\mathcal{S}$ equipped with a valuation v that satisfies heredity on atoms. A formula A is true in a substructural model $(\mathcal{S}, v)$ iff $1 \models A$. Moreover, A is valid iff it is true in every substructural model $(\mathcal{S}, v)$.*

This semantics is sufficient to show that the logic **R** is sound and complete with respect to substructural models. The motivation for introducing a ternary relation R is that it is needed for the semantics of implication: R relates the states that are making $A \rightarrow B$, A, and B hold. Although the semantics based on ternary relations has been criticized for its abstract nature, there is a number of possible intuitive reading of R, cf. [5]. One of the reading of R groups the first two components of the relation, $R[xy]z$, and can be read as "the combination of information in x and y is in z". This interpretation has been analyzed in more details by Mares [13] in terms of *situated inference*. In very abstract terms, the valuation associates situations to formulas and $s \models A$ holds whenever the information contained in situation s is relevant for A. The clause for implication states that $A \rightarrow B$ holds at s if the information contained in s combined with the information contained in r produces information t that is relevant for B. We shall focus on this reading in order to provide a concrete cognitively-oriented interpretation of ternary relations semantics.

3 Conceptual spaces

Gärdenfors [9] proposes a cognitive model of representations based on the notion of *conceptual space*. The theory of conceptual spaces is grounded on the notion of *similarity*: "[j]udgments of similarity (...) are central for a large number of cognitive processes (...) such judgments reveal the dimensions of our perceptions and their structures" ([9], p.5). *Quality dimensions*—e.g., temperature, weight, pitch, brightness—correspond to "the different ways stimuli are judged to be similar or different" ([9], p.6). They are modeled as (possibly discrete) sets of points that represent *exact* similarities between individuals. Those points represent the *qualities* of individuals: two individuals are located in the same point when they are (cognitively or empirically) indistinguishable with respect to the considered dimension, e.g., they have the same temperature, the same quality. Furthermore, dimensions have a *geometrical structure* that organizes their points according to the level of similarity between stimuli.

A set S of dimensions is *integral* if an individual located in one dimension is necessarily located also in all the other dimensions in S. For example, {*hue*, *brightness*} is integral because if an individual has a hue it necessarily has a brightness (and viceversa). A set of dimensions is *separable* if it is not integral, e.g., {*hue*, *pitch*}. In Gärdenfors's terminology, *domains* are maximal sets of integral dimensions. For example, the *hue*, *chromaticness*, and *brightness* dimensions that form the color domain {*hue*, *chromaticness*, *brightness*} are integral and separable from any other dimension. Domains are central in the work of Gärdenfors because, by means of the separability condition, they can be used to assign *properties* to individuals *independently* of other properties. For instance, in empirical terms, the weight and the color of an individual can be measured independently. The *classificatory* nature of the sensory systems is defended also by Matthen [14]. In these views, properties do not have a strong ontological connotation, they do not capture how the world is but how it appears to us through our sensory systems (or artificial sensors).[1] The properties and the conceptual spaces are understood *relativistically*: their structure depends on the underlying culture, on measurement methods and sensors (in *science*), or on interpretation of the behavior of subjects (in the case of *phenomenology*). However the determinate-determinable relation, see [19], makes sense also in this case. Fully determinate properties, i.e., maximally resolving properties according to the sensors one dispose of, are represented by points in the domain. Vice versa, determinable properties, properties that abstract from the resolution of the sensor, are represented by *regions*, i.e., sets of points in the domain. For instance, 'being scarlet' and 'being crimson' can be seen as points, while 'being red' as a region containing the previous two points. *Natural properties* are convex regions.

Conceptual spaces are defined as collections of one or more domains and *concepts* are represented as regions in conceptual spaces. They are *static* theoretical entities "in the sense that they only describe the *structure* of representations" ([9], p.31). *Natural concepts* are sets of regions in different domains "together with an assignment of salience weights to the domains and information about how the regions in different domains are correlated" ([9], p.105).

Finally, an *individual* is represented as a point in a conceptual space, a vector of coordinates in the dimensions of the space. The points of the space can then be seen as the representations of *possibilia*, the set of all the possible individuals.

4 From conceptual spaces to substructural models

Our goal is to provide a cognitive interpretation of the relevant logic **R**. More specifically, following the idea of Mares, we provide an interpretation of the substructural models of **R** (cf. Definition 2) in terms of the theory of conceptual spaces properly modified and simplified for our goal. In this section we informally present our idea while Section 5 contains the technical details.

We assume a finite and fixed number N of (disjoint) domains. The ith domain is noted $\mathbf{D}_i$. The dimensions of the domains are not relevant for our task, then, to simplify our framework, we do not consider them. Consequently, we lose the original distinction between qualities and properties and all our domains are assumed as separable from the others. In addition, (fully) determinates, originally represented by points of a domain, are here singletons. In this way, both

[1] Causation links between how the world is and how it appears to us can be considered.

the determinates and the determinables (the regions) are elements of a domain $\mathbf{D}_i$. This move simplifies the formalization and is consistent with a mereological view of domains where determinates correspond to atomic regions (see [6]). Furthermore, $\subseteq$ is the only relation between regions we consider, no topological or geometrical relations are introduced.[2] Finally, we represent the classification of objects[3] under the properties in the domains but not their *categorization* under the concepts. Actually, concepts are not needed for our goal. This may appear as an oversimplification of the original theory of conceptual spaces. However, note that (*i*) our notion of domain is perfectly aligned with the original that can be seen as a limit case of the one of concept, i.e., regions in the domains are simple concepts; (*ii*) links between domains useful to define natural concepts are modeled via *correlations* (see below); (*iii*) the basic framework introduced here can be easily modified to take into account dimensions while categorization is an extension that could underline a new kind of implication (in addition to the ones we discuss in Section 6) to be addressed in future work.

The original idea of representing individuals as vectors of points (singletons in our case) each one belonging to a different domain is too strong for our aims. This view assumes a complete knowledge about the individuals, while we are interested in the acquisition of knowledge, information, or data, about individuals. We then weaken the original theory by allowing two kinds of partial knowledge about individuals: (*i*) the exact location into a domain is not known, i.e., one can only assign a determinable property to the individual, e.g., one knows it is red, but not the exact shade of red; (*ii*) one does not have any information about a given property, one does not even know if an individual is located in a given domain, e.g., if it has a color or not. Firstly, note that in (*i*) one may consider the maximal region of a domain. That means, for instance, that one only knows that the individual is colored. Secondly, (*ii*) contemplates the case of individuals that lack some properties, i.e., individuals are not necessarily located in all the domains. For instance, abstract individuals are not in space, while holes do not weight. However, we do not represent the impossibility to be located in a domain[4] but only the lack of information (see below).

The assumption that the conditions of individuation of objects are purely conceptual has been criticized by Pylyshyn. In [16] he explores the idea that "[p]art of what it means to individuate something is to be able to keep track of its identity despite changes in its properties and location" ([16], p.33). The initial individuation and tracking of objects is not conceptual, i.e., it is not based on the classification under concepts, it is based on a lower level mechanism built into the visual system called FINST. We cannot enter here into the details of the approach. What is interesting for us is the link, provided by Pylyshyn, with the theory of *object files* [11]. One "can think of an object file as a way for informa-

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[2] Consequently, the structural relations of spaces, e.g., distances or orders, are here only used to build the taxonomy of properties. As discussed in Section 6, this structural information could be also used to represent relations among objects.

[3] From here we use 'object' and 'individual' as synonymous.

[4] That could be useful for approaching the semantics of negation.

tion to be associated with objects that are selected and indexed by the FINST mechanism. When an object first appears in view (...) a file is established for that object. Each object file has a FINST reference to the particular individual to which the information refers." ([16], p.38) The file allows us to group and maintain all the informations associated to the same individual (maybe acquired or updated through time), in particular "the one-place predicates that pertain to that object" ([16], p.39). An object file may be seen as an updatable frame-based description of an individual.[5]

Following this idea, we assume a fixed set OB of objects that are described by objects files defined as tuples $\langle a, R_1, \ldots, R_n \rangle$ where $a \in OB$ and $R_i \subseteq \mathbf{D}_i$ is a set of regions of $\mathbf{D}_i$. Firstly, object files are contextual, they depend on the chosen sets of domains and objects. Secondly, they collect all the known properties of a given object, i.e., all their known locations inside the domains. Intuitively, an object file represents the whole information about an object one has at a given stage, i.e., in an ontological perspective, the collection of *states of affairs* [3] relative to the same object. Thirdly, the R_i are sets of regions rather than simply regions. This extension of the original notion of location into domains is required to represent the process of making the acquired knowledge about an object explicit. As an illustrative example, assume that the color domain contains three subregions such that: $\texttt{scarlet} \subset \texttt{red} \subset \texttt{colored}$. In $f = \langle a, \{\texttt{scarlet}\} \rangle$ the only explicit knowledge is the scarletness of a, whereas $f' = \langle a, \{\texttt{scarlet}, \texttt{red}\} \rangle$ adds the redness of a. By looking at the structure of the color domain, the knowledge in f' was already present in f, but only in an implicit form, i.e., f' is the result of an inference process, a cognitive abstraction activity. In mathematics, one can see this situation as the introduction of a new theorem. The theorem was implicit in the theory but, by making it explicit, we add, in some sense, information.[6] Fouthly, we need to guarantee that object files contain consistent information, e.g., it is possible to have $\langle a, \{\texttt{scarlet}, \texttt{red}\} \rangle$ but not $\langle a, \{\texttt{red}, \texttt{blue}\} \rangle$ (if 'being red' and 'being blue' are disjoint). Finally, $R_i = \emptyset$ represents the total lack of information, discussed above, about the ith domain. In particular, $f = \langle a, \emptyset, \ldots, \emptyset \rangle$ represents just the existence of $a \in OB$.

A situation can be seen as a set of object files for the objects OB with respect to the domains $\mathbf{D}_1, \ldots, \mathbf{D}_{\mathrm{N}}$, i.e., as a the collection of states of affairs relative to the objects OB expressible with the same set of properties. Because the R_i in the object files may be the empty set or may represent determinable properties, in general the situations capture partial information about the objects. In particular, the situation 1 is the situation where all the object files have the form $\langle a, \emptyset, \ldots, \emptyset \rangle$, i.e., the situation 1 represents only the *terminological* knowledge.

[5] Note that we do not consider time, updating must be intended in terms of knowledge or information acquisition steps.

[6] In an empirical scenario where one disposes of instruments with different resolutions, the previous situation could be seen as the acquisition of a new measure with a coarser resolution. We do not consider this interesting observational perspective where one could also acquire new measures with identical resolution, e.g., one would be able to distinguish $\langle a, \{\texttt{scarlet}, \texttt{scarlet}\} \rangle$ from $\langle a, \{\texttt{scarlet}\} \rangle$.

Then, we model the reachability relation between situations in terms of updates of the information contained in a situation. Intuitively, given the situation s, t, and u, $Rstu$ holds when the object files in u can be obtained by means of the ones in s and t through two possible types of updating: *abstraction* and *correlation*. Abstraction generalizes conceptualization within the same domain (e.g. from scarlet to red), i.e., it relies on the $\subseteq$-structure of domains. Vice versa correlation individuates dependencies between distinct domains, for instance, it may relate colors and shapes. *Induction*, as understood by Gärdenfors, is an example of correlation: "[t]he essential role of induction is to establish *connections* among concepts or properties *from different domains*" ([9], p.211). More specifically, the "inductive process corresponds to determining *mappings* between the different domains of a space. Using such a mapping, one can then determine correlations between the regions of different domains. The correlation between two properties F and G, expressed on the symbolic level by a universal statement of the form "all Fs are Gs," would then just be a special case" ([9], p.228). We represent only the simple correlation between two properties by a pair of regions, the regions that represent these properties.

Finally, following the Routley-Meyer Semantics, the function of valuation v assigns to any atomic proposition a set of situations.

5 Conceptual spaces and relevant logic

We formally define the notions introduced in the previous section. A domain $\mathbf{D}$ is given by the set of all regions over a set of values $D = \{p_1, \ldots, p_l\}$: $\mathbf{D} = \mathcal{P}^\star(D) = \mathcal{P}(D) \setminus \emptyset$, where we exclude $\emptyset$ to avoid counterintuitive "null properties". In what follows, we fix a set of N domains $\mathbf{D}_1, \ldots, \mathbf{D}_\mathrm{N}$, denoted by $\bar{\mathbf{D}}$. We denote by r_i^1, ..., r_i^n the elements of a domain $\mathbf{D}_i$. Elements r_i^j are called *regions* of the domain. We sometimes use names for labeling regions. For instance, let $D = \{p_1, p_2, p_3\}$, then $\mathbf{D}$ has as elements regions such as $\{p_1\}$, $\{p_2\}$, and $\{p_1, p_2\}$. We may then label $\texttt{scarlet} = \{p_1\}$, $\texttt{crimson} = \{p_2\}$ and $\texttt{red} = \{p_1, p_2\}$.

Given a domain $\mathbf{D}_i$, we denote by $R_i \subseteq \mathbf{D}_i$ a set of regions in $\mathbf{D}_i$.

Definition 3 (Consistency). *We say that R_i is* consistent *iff if $R_i \neq \emptyset$, then $(\bigcap_{r \in R_i} r) \neq \emptyset$.*

Intuitively, as we will see, consistent sets of regions can be intended as non-exclusive properties that can in principle be ascribed to an object. In case the set of regions is empty, it represents the absence of information of type $\mathbf{D}_i$ concerning that object. For instance, $R_i = \{\texttt{scarlet} = \{p_1\}, \texttt{red} = \{p_1, p_2\}\}$ is consistent, since the intersection of the regions in R_i is not empty, whereas $R'_i = \{\texttt{scarlet} = \{p_1\}, \texttt{crimson} = \{p_2\}\}$ is not. That is, we can say that an object is both scarlet and red, as for instance $\texttt{scarlet} \subseteq \texttt{red}$, but we cannot say that it is both scarlet and crimson.

Moreover, we fix a set $OB = \{a_1, \ldots, a_l\}$ of objects.

Definition 4 (Object files). *An object file f_a is a vector $\langle a, R_1, \ldots, R_n \rangle$, where $a \in OB$, $R_i \subseteq \boldsymbol{D}_i$, such that each R_i is consistent.*

The set of all object files depends on the choice of $\bar{\mathbf{D}}$ and OB, so we denoted by $OBF^{OB}_{\bar{\mathbf{D}}}$.We can now introduce the definition of situation.

Definition 5 (Situation). *A situation s is a set of object files $s \subseteq OBF^{OB}_{\bar{\boldsymbol{D}}}$ such that, for every object $a \in OB$, there exist a unique object file $f_a \in s$.*

Then, we assume a number of *correlations* relating regions in different domains.

Definition 6 (Correlations). *A set of correlations COR is a set of pairs of regions (r^l_i, r^m_j), where $r^l_i \in \boldsymbol{D}_i$ and $r^m_j \in \boldsymbol{D}_j$, $i, j \in \{1, \ldots, \mathrm{N}\}$ and $i \neq j$. Moreover correlations satisfy the following conditions:*

Restricted transitivity *if $(r^l_i, r^m_j) \in COR$, $(r^m_j, r^n_h) \in COR$, and $h \neq j$, then $(r^l_i, r^n_h) \in COR$.*

Correlation composition *if $(r^l_i, r^m_j) \in COR$ and $r^h_i \subseteq r^l_i$, then $(r^h_i, r^m_j) \in COR$; if $(r^l_i, r^m_j) \in COR$ and $r^m_j \subseteq r^h_j$, then $(r^l_i, r^h_j) \in COR$.*

Restricted transitivity states that if we can connect two regions in a number of steps, we can also connect them by composing the correlations in one single step. The condition $h \neq j$ in the restricted transitivity excludes that we end up relating regions of the same domain. For instance, it prevents passing from (`scarlet`, `round`) and (`round`, `crimson`) to (`scarlet`, `crimson`). The rules for correlation composition state that if we correlate a concept with another, the correlation applies also to the subconcept of the first one and to super-concept of the second one. For instance, if we say that red things are round, we also say that scarlet things are round. We do not put any further consistency constraint on correlations. The reason is that correlations are intended to represent factual, but not necessarily correct, mappings between concepts. For instance, we do not exclude from COR correlations that can end up in inconsistent outcomes, e.g. (`round`, `scarlet`) and (`round`, `crimson`). The point is that correlations express matters of fact, thus they are falsifiable and in principle revisable. By contrast, conceptual information is fixed and non-revisable.

We turn now to the interpretation of the ternary relation R in our setting. Intuitively, situations are related if they are reachable by means of an abstraction move or by means of a correlation link. Denote by f^s_a the (unique) object file f_a in situation s. Moreover, denote by $R^s_{a,i}$ the set of regions of $\mathbf{D}_i$ that in situation s are associated to object a. We are ready now to present our interpretation of the ternary relation in terms of reachability of situations.

Definition 7 (Reachability of situations). *Let u, t and s situations in $OBF^{OB}_{\bar{\boldsymbol{D}}}$. The situation u is reachable from t given s, i.e., $Rstu$, iff:*

R1. for all $a \in OB$, for all $R^u_{a,i}$ then $R^u_{a,i} \supseteq (R^s_{a,i} \cup R^t_{a,i})$; i.e., all the data in s and t are imported in u;

R2. for all $a \in OB$, for all $r \in R^u_{a,i} \backslash (R^s_{a,i} \cup R^t_{a,i})$, r is obtained in one of the two following ways:

Abstraction *there exists $r' \in R^s_{a,i} \cup R^t_{a,i}$ such that $r' \subseteq r$;*

Correlation *there exists* $r' \in R^s_{a,j} \cup R^t_{a,h}$ *such that* $(r', r) \in COR$.

$Rstu$ imposes that the whole information in u is derived (by using conceptual knowledge or correlations) from the one in s and the one in t. R1 entails that the regions in s and t are preserved in u. R2 shows that all the new regions in u are derived from the ones in s and t by abstraction or by correlation. Note that, in principle, a situation could be updated through abstraction and correlation into something that is not a situation, i.e., into a set of inconsistent object files. For instance, suppose that $(\texttt{round}, \texttt{crimson}) \in COR$ and that `scarlet` and `crimson` are disjoint. Suppose $\bar{\mathbf{D}}$ contains just two domains, e.g. colors and shapes. Thus, a situation s that contains $\langle a, \{\texttt{scarlet}, \texttt{red}\}, \{\texttt{round}\}\rangle$ can be updated, by means of the correlation $(\texttt{round}, \texttt{crimson})$, to a set of object files that contains $\langle a, \{\texttt{scarlet}, \texttt{red}, \texttt{crimson}\}, \{\texttt{round}\}\rangle$, which violates consistency of the sets of regions that is required for object files. Since we are assuming that R is defined on situations, i.e. sets of object files with consistent R_i-sets, the case above is excluded. This point shows a significant difference between abstraction and correlation: abstraction guarantees consistency of the update, whereas correlation does not. This reflects the distinction between conceptual and factual knowledge. Once the conceptual relations are set and we have assumed that they are consistent, by abstraction we can only generalize on given data. By contrast, correlations introduce new data that may be inconsistent with previous ones.

We define the following relation of *consistency* between situations (Cst)

Definition 8 (Consistent situations Cst). *The two situations s and t are consistent, noted by Cst, iff:*

C1. for $i \in \{1, \ldots, \mathrm{N}\}$, $R^s_{a,i} \cup R^t_{a,i}$ *is consistent* (*cf. Definition 3*)

By means of Definition 7, we can infer that, if a situation u is reachable from t given s, then u is consistent both with s and with t and s is consistent with t.

Proposition 1. *If $Rstu$, then Csu, Ctu, and Cst.*

Proof. Assume $Rstu$, that entails by R1 that for every i and every object a, $R^s_{a,i} \cup R^t_{a,i} \subseteq R^u_{a,i}$. Thus, since $R^u_{a,i}$ is consistent by definition, then $R^s_{a,i} \cup R^t_{a,i}$ is consistent, so Cst. The other cases follows by noticing that $R^s_{a,i} \subseteq R^u_{a,i}$ and $R^t_{a,i} \subseteq R^u_{a,i}$. □

We conclude this paragraph by providing an interpretation of the element 1 of the substructural model. We define 1 as the situation in which we have no information about any object, i.e., $1 := \{\langle a, \emptyset, \ldots, \emptyset\rangle \mid a \in OB\}$. Every $\langle a, \emptyset, \ldots, \emptyset\rangle$ is an object file, that is, it satisfies consistency. Moreover 1 is a situation, since for every object, there exist a unique object file in 1.

5.1 Conceptual spaces as models of R

We can now show that our view of situations provides a model of relevant logic.

Definition 9 (Conceptual substructural model). *A conceptual substructural model is given by* $(\langle \mathcal{S}, COR, R, 1\rangle, v)$*, where* $\langle \mathcal{S}, COR, R, 1\rangle$ *is a conceptual substructural frame:* $\mathcal{S}$ *is a set of situations defined wrt. a domain* $\bar{\boldsymbol{D}}$ *and a set of objects* OB*,* COR *is a set of correlation between regions of* $\bar{\boldsymbol{D}}$*,* $R \subseteq \mathcal{S}^3$ *is a reachability relation, and* $1 := \{\langle a, \emptyset, \dots, \emptyset\rangle \mid a \in OB\}$*. Moreover,* v *is a valuation that associates to atoms sets of situations, i.e.,* $v : Atom \to \mathcal{P}(\mathcal{S})$ *such that heredity holds.*

We only need to show that R and 1 satisfy the axioms of Definition 1.

Proposition 2. *The reachability of situations* R *satisfies axioms A1, A2, A3, A4, and A5 of Definition 1.*

Proof. We only show the details of the representative cases.
A1: $R1ss$. R1 trivially holds. R2 holds because $R^s_{a,i} \setminus (R^1_{a,i} \cup R^s_{a,i}) = \emptyset$.
A2: If $Rstu$, then $Rtsu$. It is sufficient to notice that the definition of R is symmetric wrt. $R^s_{a,i}$ and $R^t_{a,i}$.
A3: If $R^2(st)uw$, then $R^2(su)tw$. We need to show that if there exists an x such that $Rstx$ and $Rxuw$, then there exists a y such that $Rsuy$ and $Rytw$. Assume that there exists an x such that $Rstx$ and $Rxuw$.
We show that there is a y such that $Rsuy$ and $Rytw$. We set for every a and i, $R^y_{a,i} = R^s_{a,i} \cup R^u_{a,i}$.
Firstly, we show that $Rsuy$. We have that $R^s_{a,i} \cup R^u_{a,i} \subseteq R^y_{a,i} = R^s_{a,i} \cup R^u_{a,i}$, thus R1 is fine. Since there is no other regions in $R^y_{a,i}$, we can conclude that R2 is also satisfied. Hence, $Rsuy$.
Then, we have to show $Rytw$. By assumption, $R^x_{a,i} \cup R^u_{a,i} \subseteq R^w_{a,i}$, thus we can deduce $R^y_{a,i} \cup R^t_{a,i} \subseteq R^w_{a,i}$. So R1 is satisfied.
Suppose now that there is an $r \in R^w_{a,i} \setminus R^y_{a,i} \cup R^t_{a,i}$, that is $r \in R^w_{a,i} \setminus R^s_{a,i} \cup R^u_{a.i} \cup R^t_{a,i}$. Since by assumption $Rxuw$, every region r in w is obtained by abstraction or correlation from regions in x or u. If r is obtained by abstraction or correlation from a region in $R^u_{a,i}$, then we are done, since $R^u_{a,i} \subseteq R^y_{a,i}$. If r is obtained from regions in x, then, by assumption $Rstx$, so r is obtained from regions that are either in s or t. We approach the following cases:
(i) r is obtained by correlation from an $r' \in R^x_{a,i}$ and r' is obtained from correlation from an $r'' \in R^s_{a,i}$. This means that $(r', r), (r'', r') \in COR$, thus, by restricted transitivity, $(r'', r) \in COR$. Therefore, if $r \in R^w_{a,i} \setminus R^s_{a,i} \cup R^u_{a.i} \cup R^t_{a,i}$, then there is an $r'' \in R^s_{a,i}$ such that $(r'', r) \in COR$, thus we conclude;
(ii) r is obtained by correlation from an $r' \in R^x_{a,i}$ and r' is obtained by abstraction from an $r'' \in R^s_{a,i}$. This means that $r'' \subseteq r'$ and $(r', r) \in COR$, thus, by the first rule of correlation compoistion, $(r'', r) \in COR$. Therefore, for $r \in R^w_{a,i} \setminus R^s_{a,i} \cup R^u_{a.i} \cup R^t_{a,i}$, there is an $r'' \in R^s_{a,i}$ such that $(r'', r) \in COR$ and we conclude again;
(iii) r is obtained by abstraction from r' in x and r' is obtained by abstraction from r'' in s. Then r can be obtained by abstraction from r'' and we are done;
(iv) r is obtained by abstraction from r' in x and r' is obtained by correlation $(r'', r') \in COR$ from r'' in s. In this case, $r' \subset r$ and $(r'', r') \in COR$, thus by the second rule of correlation composition we infer $(r'', r) \in COR$ and we conclude.

Therefore, also R2 is satisfied, therefore $Rytw$.
A4: $Rsss$ holds, since R1 trivially holds and $R^s_{a,i} \setminus (R^s_{a,i} \cup R^s_{a,i}) = \emptyset$.
A5: If $Rstu$ and $w \leq s$, then $Rwtu$. Recall that $w \leq s$ is defined by $R1ws$. We show only the following case. Suppose $r \in R^u_{a,i} \setminus R^w_{a,i} \cup R^t_{a,i}$ and that r is obtained by means of correlation from r'. In case $r' \in R^w_{a,i} \cup R^t_{a,i}$, we are done. Otherwise, by assumption $r' \in R^s_{a,i}$ and $(r, r') \in COR$. Since $R1ws$, there are two cases. Firstly, there exists r'' such that $(r'', r') \in COR$. By restricted transitivity, we conclude that for $r \in R^u_{a,i} \setminus R^w_{a,i} \cup R^t_{a,i}$, there exists an r'' in $R^w_{a,i}$ such that $(r'', r) \in COR$. Secondly, r' is obtained by abstraction from r'' in $R^w_{a,i}$, in this case by correlation composition, we conclude. □

It is important to notice that the provided interpretation in terms of situations does not trivialize the substructural model, namely R does not provide a model of intuitionistic or classical implication. To see that, we show that monotonicity does not hold in conceptual substructural models. In axiomatic terms, monotonicity corresponds to the validity of $A \rightarrow (B \rightarrow A)$. In semantic terms, it corresponds to the following constraint on the ternary relation [7]:

f1 $Rstu \Rightarrow R1su$

Consider a simple example where $s = \{\langle a, \emptyset\rangle, \langle b, \{\texttt{scarlet}\}\rangle\}$, $t = \{\langle a, \{\texttt{scarlet}\}\rangle, \langle b, \emptyset\rangle\}$, and $u = \{\langle a, \{\texttt{scarlet}, \texttt{red}\}\rangle, \langle b, \{\texttt{scarlet}\}\rangle\}$. In this case, although $Rstu$, neither $R1su$ nor $R1tu$ hold, i.e., both the information in s and t is needed for u. Therefore, (f1) does not hold in every conceptual substrctural model, thus $A \rightarrow (B \rightarrow A)$ is not valid.

6 Conclusions and future work

We presented a concrete instantiation of the ternary relation model of the relevant logic **R** that is grounded on the framework of conceptual spaces.Our instantiation of the Routley-Meyer semantics provides a number of reasons to interpret relevant implication in terms of cognitively aware updates of knowledge. Besides the logical contribution, we believe that both the notion of situation and the one of reachability between situations provide a useful framework for separating conceptual and factual knowledge and for modeling knowledge acquisition. However, the cognitive plausibility of the interpretation of the inferential mechanism we proposed still lacks an empirical assessment.

Future work concerns two directions. Firstly, notice that the proposed framework provides an interpretation only to atomic propositions that reduce to the assignment of a (unary) property to an object, it does not consider relations among objects. The extension to relations definable in terms of relations among intrinsic properties of the relata is quite trivial.[7] Using conceptual spaces, this kind of relations can be represented by means of *higher level properties* (see [9,

[7] Even though one has to decide whether relational information is encoded in the objects files—e.g., if $REL(a, b)$ holds then one needs to add this information in both the object-file relative to a and b—or outside them.

sect.3.10.1]). For instance, suppose to have the dimension *length* structured by the order relation $\leq$. The relation *shorter than* can be represented by a region in the space of the pairs of length-values, i.e. the region of all pairs (l_1, l_2) such that $l_1 \leq l_2$. Thus, an object x is shorter than an object y if the pair (length of x, length of y) belongs to this region. Gärdenfors seems to suggest that this approach is general enough to represent all the (binary) relations: "[a] relation between two objects can be seen as a simple case of a *pattern* of the location of the objects along a particular quality dimension" [9, p.93]. However, some structural relations, e.g., part-whole relations, seem to require really complex spaces founded on several quality dimensions (see [17]). More importantly, it is not clear to us how some relations like *eat* or *married to* can be reduced to intrinsic properties of relata. Similarly for *relational categories* [10], i.e., properties that are defined in relational terms, e.g., a carnivore is an animal that eats meat.

Secondly, in Definition 7, we have distinguished two possible ways of updating the information contained in a situation: abstraction and correlation. We have suggested that, intuitively, they correspond to two distinct types of processes: the first abstracts from already given data, the second allows to *indirectly* discover new data, a sort of indirect measurements. Contrast the following sentences:

i. "If a is scarlet, then a is red" and

ii. "If a is scarlet, then a is round."

In our model, (i) updates a situation s that contains, let say, $\langle a, \{\texttt{scarlet}\}, \emptyset\rangle$ into a situation t that contains $\langle a, \{\texttt{scarlet}, \texttt{red}\}, \emptyset\rangle$ whereas (ii) is an update from s to a situation t' that contains $\langle a, \{\texttt{scarlet}\}, \{\texttt{round}\}\rangle$. Both these updates add information that was implicit, but they qualitatively differ because the update in (i) impacts the same domain while the one in (ii) impacts a different domain. Furthermore, our intuition is that (i) holds just in virtue of 'the scarletness of a', while (ii) holds in virtue of both 'the scarletness of a' and 'the roundness of a' (assuming that 'being scarlet' and 'being round' are both fully determinate properties). In terms of truth-makers (see [4]) this means that the two propositions 'a is scarlet' and 'a is red' share the same truth-maker ('the scarletness of a'). By contrast, the two propositions 'a is scarlet' and 'a is round' need two different truth-makers, i.e., only the second inference reveals the existence of an implicit truth-maker. This would suggest that the first kind of reasoning, the abstraction, is a purely mental process that does not need verification. By contrast, the second kind of reasoning, the correlation, needs additional validation in terms of truth-makers. Actually this provides a partial justification of the asymmetry between the required consistency of the conceptual knowledge vs. the possibility to have inconsistent correlations. An interesting question is whether it is possible to distinguish the two process in terms of inferential patterns, that is, we ask whether it is meaningful to define two kinds of implications, one corresponding to the sole updating by abstraction, and one corresponding to updating by correlation. We leave for future work the axiomatization of these two types of implications that, in our framework, can be characterized by two distinct reachability relations: one that only permits updates by abstraction, the other that only permits updates by correlations.

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