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Modelling Multilateral Negotiation in Linear Logic

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Abstract. We show how to embed a framework for multilateral negotiation, in which a group of agents implement a sequence of deals concerning the exchange of a number of resources, into linear logic. In this model, multisets of goods, allocations of resources, preferences of agents, and deals are all modelled as formulas of linear logic. Whether or not a proposed deal is rational, given the preferences of the agents concerned, reduces to a question of provability, as does the question of whether there exists a sequence of deals leading to an allocation with certain desirable properties, such as maximising social welfare. Thus, linear logic provides a formal basis for modelling convergence properties in distributed resource allocation.

1 INTRODUCTION

AI applications often involve some form of multiagent resource allocation [1]: e.g., in cooperative problem solving, we need to find an allocation of resources to agents that will allow each agent to complete the tasks she has been assigned; in the context of electronic commerce applications, the system objectives will often be defined in terms of properties of the allocations of resources that are being negotiated. Studies of resource allocation in AI may range from the design of negotiation strategies, over the game-theoretical analysis of allocation problems, to the complexity-theoretic study of relevant optimisation problems. While some of the work in the field is very pragmatic in nature, and rightly so, at one end of the spectrum, fundamental research in AI should (and does) develop frameworks for the precise representation and formal study of systems for multiagent resource allocation and negotiation. Logic suggests itself as a tool for this purpose, and there have been a number of contributions of this kind [4, 5, 8, 11, 12, 15, 16], some of which we shall review below.

In this paper, we show how to embed a framework for *distributed resource allocation* [2, 3], in which a group of agents implement a sequence of multilateral deals concerning the exchange of a number of resources, into *linear logic* [6]. In our model, multisets of goods, allocations of resources, preferences of agents, and deals are all modelled as formulas of linear logic. Whether or not a proposed deal is rational, given the preferences of the agents concerned, reduces to a question of provability, as does the question of whether there exists a sequence of deals leading to an allocation with certain desirable properties, such as maximising social welfare.

There have been a number of previous contributions that use different kinds of logical frameworks to model a variety of aspects of negotiation in multiagent systems. In an early contribution to the field, Fisher [5] shows how to reduce the problem of constructing coherent negotiation dialogues to (distributed) theorem-proving. Indeed, most work on logic-based approaches to negotiation and resource allocation deals with this domain of "symbolic negotiation". Sadri et al. [16], for instance, do so in the framework of abductive logic programming. Several authors have recognised that, due to its resource-sensitive nature, linear logic is particularly suited to modelling resource allocation problems [8, 11, 15]. In particular, as far as modelling the complex preferences of agents over bundles of resources are concerned, we build directly on our recent work [15], in which we have developed bidding languages for multi-unit combinatorial auctions based on linear logic.

Two contributions on logic-based approaches to resource allocation relate to the same kind of resource allocation framework we shall be working with here: Endriss and Pacuit [4] develop a modal logic to study the convergence problem in distributed resource allocation; and Leite et al. [12] show how to map the problem of finding an allocation that is socially optimal (for a wide variety of fairness and efficiency criteria) into the framework of answer-set programming.

The remainder of the paper is organised as follows. In Section 2, we briefly review the distributed resource allocation framework we shall adopt and Section 3 covers the necessary background on linear logic. In Section 4, we define classes of valuations on multisets of goods and in Section 5 we model social welfare of allocations. In Section 6, we present a language to express deals and in Section 7 we define the relevant notions of rationality and we prove the results connecting deals and social welfare. Section 8 concludes.

2 MULTIAGENT RESOURCE ALLOCATION

In this section, we briefly review the the framework of the *distributed approach* to multiagent resource allocation [2, 3]. (There are two differences between the cited literature and our presentation here: we allow for resources to be available in *multiple units* and we restrict utility values and prices to *integers*.) In the body of the paper, we will then show how to model this framework in linear logic.

Let $\mathcal{N} = \{1, \ldots, n\}$ be a finite set of *agents* and let $\mathcal{M}$ be a finite multiset of *resources*. We denote the set of *types of resources* in $\mathcal{M}$ by $\mathcal{A}$ (as these will be the *atoms* of our logical language). An *allocation* is a mapping $\alpha : \mathcal{M} \to \mathcal{N} \cup \{*\}$ from resources to agents; indicating for each item who receives it or whether it does not get allocated at all $(*)$. $A_i = \alpha^{-1}(i)$ is the multiset of resources given to agent $i \in \mathcal{N}$. We will refer to allocations both in terms of α and A. A *deal* takes us from one allocation to the next; i.e., we can think of it as a pair of allocations. Note that there are no restrictions as to the number of agents or resources involved in a single deal. Of special interest are structurally simple deals: for instance, *1-deals* are deals involving the reassignment of a single resource only.

Each agent $i \in \mathcal{N}$ is equipped with a *valuation function* $v_i : \mathcal{P}(\mathcal{M}) \to \mathbb{N}$ (including 0), mapping multisets of resources she may receive to their value. The valuations of individual agents can be used to define what constitutes a desirable allocation. We will concentrate on two economic efficiency criteria [1]: (1) the (utilitarian) *social*

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welfare of an allocation A is defined as $sw_u(A) = \sum_{i \in \mathcal{N}} v_i(A_i)$ and we shall be interested in finding allocations that maximise social welfare; (2) an allocation is *Pareto optimal* if no other allocation gives higher valuation to some agents without giving less to any of the others (this is a considerably less demanding criterion).

What kinds of allocations can be reached from a given initial allocation depends on the range of deals we permit. A deal is called *individually rational* if it is possible to arrange side payments for the agents involved such that for each agent her gain in valuation outweighs her loss in money (or her gain in money outweighs her loss in valuation). The payments of all agents need to add up to 0.

It is possible to show, rather surprisingly, that any sequence of individually rational deals will always converge to an allocation with maximal social welfare [17]. The proof of this result crucially relies on the insight that increases in social welfare are in fact *equivalent* to individual rationality [3]. For certain restricted classes of valuation functions it is furthermore possible to prove convergence by means of structurally simple deals [2].

To be precise, social welfare increase is equivalent to individual rationality in case valuation functions are real-valued. In this paper, we assume that valuation functions are integer-valued; we will adapt the definition of individual rationality accordingly and consequently obtain a slightly different (namely, stronger) convergence result.

3 LINEAR LOGIC

We briefly present some essential features of linear logic (LL); for full details, we refer to Girard [7] and Troelstra [18]. LL provides a resource-sensitive account of proofs by controlling the amount of formulas actually used. In classical logic, sequents are defined as $\Gamma \vdash \Delta$, where Γ and Δ are sets. According to the structural rules of the Gentzen sequent calculus, we can for example delete a second copy of a formula. In LL, structural rules are dropped, so Γ and Δ are to be considered multisets of formulas. The absence structural rules has important consequences on the logical connectives we can define.

Given a set of positive atoms $\mathcal{A}$, the language of LL is defined as follows (where $p \in \mathcal{A}$): L ::=

$$p \mid \mathbf{1} \mid \bot \mid \top \mid \mathbf{0} \mid L^{\bot} \mid L \otimes L \mid L \parr L \mid L \oplus L \mid L \,\&\, L \mid !L \mid ?L$$

Linear negation $(\cdot)^{\bot}$ is involutive and each formula in LL can be transformed into an equivalent formula where negation occurs only at the atomic level. The conjunction $A \otimes B$ ("tensor") means that we have exactly one copy of A and one copy of B, no more no less. Thus, e.g., $A \otimes B \nvdash A$. We might say that in order to sell A and B, we need someone who buys A and B, not just a buyer for A. We will not directly use the disjunction $A \parr B$ ("par"); rather we use linear implication: $A \multimap B := A^{\bot} \parr B$. Linear implication is a form of deal: "for A, I sell you B". The additive conjunction $A \,\&\, B$ ("with") introduces a form of choice: we have one of A and B and we can choose which one. For example, $A \,\&\, B \vdash A$, but we do not have them both: $A \,\&\, B \nvdash A \otimes B$. The additive disjunction $A \oplus B$ ("plus") means that we have one of A and B, but we cannot choose, e.g., $A \vdash A \oplus B$ but $A \oplus B \nvdash A \,\&\, B$. The exponentials $!A$ and $?A$ reintroduce structural rules in a local way: !-formulas licence contraction and weakening on the lefthand side of $\vdash$; ?-formulas licence structural rules on the right. Intuitively, exponential formulas can be copied and erased; they are relieved from their linear status and can be treated as elements of sets again.

We will use the intuitionistic version of LL (ILL), obtained by restricting the righthand side of the sequent to a single formula, so we will not use ? and $\parr$. In fact, we will mostly use ILL augmented with the global weakening rule (W), also known as *affine logic* [10, 18]:

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \mathrm{W}$$

The rules of the sequent calculus for ILL are shown in Table 1 [18].

We can restrict attention to the following fragments: *intuitionistic multiplicative linear logic* (IMLL) using only $\otimes$ and $\multimap$; *intuitionistic multiplicative additive linear logic* (IMALL) using only $\otimes$, $\multimap$, $\&$ and $\oplus$; and *Horn linear logic* (HLL). In the latter, sequents must be of the form $X, \Gamma \vdash Y$ [9], where X and Y are tensors of positive atoms, and Γ is one of the following (with X_i, Y_i being tensors of positive atoms): (i) $(X_1 \multimap Y_1) \otimes \cdots \otimes (X_n \multimap Y_n)$, (ii) $(X_1 \multimap Y_1) \,\&\, \cdots \,\&\, (X_n \multimap Y_n)$.

For these fragments, we have the following proof-search complexity results. MLL is NP-complete and so is MLL with full weakening (W) [13]. The same results apply for the intuitionistic versions. HLL is NP-complete, and so is HLL + W [9]. MALL and IMALL are PSPACE-complete [14].

$$\frac{}{A \vdash A}\ \mathrm{ax} \qquad \frac{\Gamma, A \vdash C \qquad \Gamma' \vdash A}{\Gamma, \Gamma' \vdash C}\ \mathrm{cut}$$

MULTIPLICATIVES

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes\mathrm{L} \qquad \frac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}\ \otimes\mathrm{R}$$

$$\frac{\Gamma \vdash A \qquad \Gamma', B \vdash C}{\Gamma', \Gamma, A \multimap B \vdash C}\ \multimap\mathrm{L} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap\mathrm{R}$$

$$\frac{\Gamma \vdash C}{\Gamma, \mathbf{1} \vdash C}\ \mathbf{1}\mathrm{L} \qquad \frac{}{\vdash \mathbf{1}}\ \mathbf{1}\mathrm{R}$$

ADDITIVES

$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_0 \& A_1 \vdash C}\ \&\mathrm{L} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}\ \&\mathrm{R}$$

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ \oplus\mathrm{L} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \oplus A_1}\ \oplus\mathrm{R}$$

$$\frac{}{\Gamma, \mathbf{0} \vdash C}\ \mathbf{0}\mathrm{L} \qquad \frac{}{\Gamma \vdash \top}\ \top\mathrm{R}$$

EXPONENTIALS

$$\frac{\Gamma, A \vdash C}{\Gamma, !A \vdash C}\ !\mathrm{L} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ !\mathrm{R}$$

STRUCTURAL RULES

$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\ \mathrm{P} \qquad \frac{\Gamma, !A, !A, \vdash C}{\Gamma, !A \vdash C}\ !\mathrm{C} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\ !\mathrm{W}$$

Table 1. Sequent Calculus for Intuitionistic LL

4 VALUATIONS

Recall that $\mathcal{M}$ is a finite multisets of resources and that $\mathcal{A}$ is the set of distinct (types of) resources in $\mathcal{M}$. We call $\mathcal{A}$ the *atoms*, and we will build a logical language based on these atoms. In this section, we will show how to represent different classes of valuation functions over the powerset of $\mathcal{M}$ in LL.

There is an isomorphism between multisets and tensor formulas of atoms (up to associativity and commutativity):

$$\{m_1, \ldots, m_k\} \;\cong\; m_1 \otimes \cdots \otimes m_k$$

Thus, we can represent each subset $X \subseteq \mathcal{M}$ as a tensor product. Moreover, if $M \cong A$ and $N \cong B$, then the (disjoint) union of M and N is isomorphic to $A \otimes B$.

We want to define languages to encode *valuation functions* $v : \mathcal{P}(\mathcal{M}) \to \mathbb{N}$ (including 0), mapping subsets of $\mathcal{M}$ to values. To model values, assume a finite set of distinct weight atoms $\mathcal{W} = \{w_1, ..., w_p\}$. In fact, we will use just one weight atom u. We write u^k for the tensor product $u \otimes \cdots \otimes u$ (k times). To associate weights with numbers, we define a function $val : \mathcal{W} \to \mathbb{N}$, with $val(u) = 1$. Let $\mathcal{W}^{\otimes}$ be the set of all finite tensor products of atoms in $\mathcal{W}$, modulo commutativity (including the "empty" product $\mathbf{1}$). That is, $\mathcal{W}^{\otimes} = \{\mathbf{1}, w_1, w_2, w_1 \otimes w_2, \ldots\}$. We extend val to $\mathcal{W}^{\otimes}$ by stipulating $val(\mathbf{1}) = 0$ and $val(\varphi \otimes \psi) = val(\varphi) + val(\psi)$. In particular, this means that $val(u^k) = k$.

We first define *atomic valuations*, which specify which value w is associated to a multiset M.

Definition 1. *An atomic valuation is a formula $M \multimap w$, where M is a tensor product of atoms in $\mathcal{A}$ and $w \in \mathcal{W}$.*

We will consider various languages to express valuations; they all share the same definition of generated function.

Definition 2. *Every formula φ in any of our languages generates a valuation v_φ mapping multisets $X \subseteq \mathcal{M}$ to their value:*

$$v_\varphi(X) = \max\{val(w') \mid w' \in \mathcal{W}^{\otimes} \text{ and } X, \varphi \vdash w'\}$$

In the case of atomic valuations $\varphi = (M \multimap w)$, this simply says that $v_{M \multimap w}(X) = w$ whenever X is equal to (or a superset of) the multiset isomorphic to M, and $v_{M \multimap w}(X) = 0$ otherwise.

In case the only weight atom used is u, i.e., if $\mathcal{W} = \{u\}$, then Definition 2 can be restated as follows:[2]

$$v_\varphi(X) = \max\{k \mid X, \varphi \vdash u^k\}$$

Now, consider the following classes of valuations.

Definition 3. *Let $v : \mathcal{P}(M) \to \mathbb{N}$; we say that:*
- *v is monotonic if $M_1 \subseteq M_2$ implies $v(M_1) \leq v(M_2)$.*
- *v is additive if $v(M) = \sum_{m \in M} v(\{m\})$ for all $M \subseteq \mathcal{M}$.*
- *v is dichotomous if $v(M) = 0$ or $v(M) = 1$ for all $M \subseteq \mathcal{M}$*
- *v is 0-1 if it is add. and $v(\{r\}) = 0$ or $v(\{r\}) = 1$ for all $r \in \mathcal{M}$.*

We can define the following languages to encode valuations. They are obtained by restricting the language of LL.

VAL $:= X \multimap w \mid$ VAL & VAL
ADD $:= a \multimap w \mid$ ADD $\otimes$ ADD
DIC $:= X \multimap \mathbf{1} \mid X \multimap u \mid$ DIC & DIC
01 $:= a \multimap \mathbf{1} \mid a \multimap u \mid 01 \otimes 01$

The class of all valuations from $\mathcal{P}(\mathcal{M})$ to $\mathbb{N}$ can be generated by the formula $\&_{X \subseteq \mathcal{M}}(X \multimap w)$, which encodes the graph of the function v.

[2] If we define $u^0 = \mathbf{1}$, by weakening (which represents monotonicity), from $\vdash \mathbf{1}$ we get $\Gamma \vdash \mathbf{1}$, for any Γ. So every valuation produces at least u^0, since it will always be satisfied by any allocation (also by allocating nothing), e.g., $p, p \otimes q \multimap u^k \vdash \mathbf{1}$ will be provable.

Proposition 4. *The following facts hold:*
- VAL *generates all valuations $v : \mathcal{P}(M) \to \mathbb{N}$.*
- VAL *under weakening generates all monotonic valuations and only those.*

For a proof, we refer to [15]. The next result is immediate, and we omit its proof for lack of space.

Proposition 5. *The following facts hold:*
- DIC *generates all dichotomous valuations and only those.*
- 01 *generates all 0-1 valuations and only those.*

Moreover, we have:

Proposition 6. ADD *generates all additive valuations and only those.*

Proof. Consider any formula ADD $= (a_1 \multimap w_1) \otimes \cdots \otimes (a_p \multimap w_p)$. We prove that it generates an additive function. Let $u^k = v_{\text{ADD}}(M)$. Since a singleton can satisfy at most one implication in ADD, for any $m \in M$, we have $v_{\text{ADD}}(\{m\}) = w_i$ if $m = a_i$, otherwise we have $v_{\text{ADD}}(\{m\}) = 0$. If we take all the m_i providing non-zero utility, we can build the following proof (by $\otimes R$ and $\otimes L$):

$$\frac{m_1, a_{l_1} \multimap w_{l_1} \vdash w_{l_1} \qquad \ldots \qquad m_k, a_{l_k} \multimap w_{l_k} \vdash w_{l_k}}{m_1, \ldots, m_k, (a_{l_1} \multimap w_{l_1}) \otimes \cdots \otimes (a_{l_k} \multimap w_{l_k}) \vdash w_{l_1} \otimes \cdots \otimes w_{l_k}}$$

If $(a_{l_1} \multimap w_{l_1}) \otimes \cdots \otimes (a_{l_k} \multimap w_{l_k})$ is not equal to ADD, then we can use weakening and $\otimes L$ to get the full formula. Thus M, ADD proves $\sum_{m \in M}\{v(\{m\}) \mid v(\{m\}) \neq 0\}$. That the value actually equals the maximum follows, since all the non-zero elements are there.

Conversely, take an additive function v, define the additive formula as follows. For all $m \in \mathcal{M}$, consider $v(\{m\}) = h_m$. If $h \neq 0$, write $(m \multimap h_m)$, otherwise write nothing. The expression $\otimes_{m \in \mathcal{M}}(m \multimap h_m)$ generates v. □

5 ALLOCATIONS

In this section, we show how to represent allocations (and their properties) in LL. We will model an allocation producing a certain social welfare as a proof for a particular LL sequent. Recall that $\mathcal{N} = \{1, ..., n\}$ is the set of agents. We add to the set of atoms $\mathcal{A} = \{p_1, \ldots, p_m\}$ all atoms p_i^j, with $i \leq m$ and $j \leq n$, to express that the good p_i is allocated to the individual j. From now on, we will assume that valuations are defined using these indexed names of resources; agents must express their preferences using the set of atoms $\mathcal{A}^j = \{p_1^j, \ldots, p_m^j\}$.

To express that each (copy of a) resource may be given to any of the agents (but not to more than one), we use the following formula:

$$\text{MAP} \;:=\; \bigotimes_{p \in \mathcal{A}} [\&_{j \in \mathcal{N}}(p \multimap p^j)]^{\mathcal{M}(p)} \qquad (1)$$

Following [15], we now define the concept of *allocation sequent*, which encodes a feasible allocation returning a particular social welfare. We take $\mathcal{M}$ and $\mathcal{N}$ to be fixed, and MAP to be defined accordingly.

Definition 7. *The allocation sequent for value k and valuations* VAL$_1, \ldots,$ VAL$_n$ *is defined as the following LL sequent:*

$$\mathcal{M}, \text{MAP}, \text{VAL}_1, \ldots, \text{VAL}_n \vdash u^k$$

The following proposition states the relationship between proofs and allocations.

Proposition 8. *Given n formula in a given class of formulas* VAL, *every allocation A with value k provides a proof π of an allocation sequent for k, and vice versa, every proof π of an allocation sequent for k provides an allocation α with value k.*

For the proof we refer to [15]. Given an allocation sequent, we can read the allocation A considering the atoms that have been actually used in the proof.

Example 9. *Consider the following allocation sequent:*

$$p, q, r, \overbrace{p \multimap p^1 \,\&\, p \multimap p^2, q \multimap q^1 \,\&\, q \multimap q^2, r \multimap r^1 \,\&\, r \multimap r^2}^{\text{MAP}},$$
$$p^2 \otimes q^2 \multimap w, r^1 \multimap v \vdash w \otimes v$$

We can retrieve A from the proof π of the sequent, which contains the following steps:

$$\frac{p^2, q^2 \vdash p^2 \otimes q^2 \qquad w \vdash w}{p^2, q^2, p^2 \otimes q^2 \multimap w \vdash w} \qquad \frac{r^1 \vdash r^1 \qquad v \vdash v}{r^1, r^1 \multimap v \vdash v}$$

So the multiset of allocated goods, $A = \{p^2, q^2, r^1\}$, can be read from the axioms in π.

Define $A_i \subseteq A$ to be the multiset of atoms allocated to agent i: $\{p^i \mid p^i \in A\}$. We can state the definitions of social welfare within our framework as follows.

Definition 10 (Utilitarian social welfare)**.**

$$sw_u(A) = \max\{k \mid A, \text{VAL}_1, \ldots, \text{VAL}_n \vdash u^k\}$$

We can consider a particular proof π of an allocation sequent and define the value of the allocation in that proof as $sw_u^\pi(A) = u^k$, where $A, \text{VAL}_1, \ldots, \text{VAL}_n \vdash u^k$ is in π. The value of the allocation for a certain agent i is given by: $u_i^\pi(A) = w_i$, where $A, \text{VAL}_i \vdash w_i$ is in π. So, for example, the utilitarian social welfare of a given allocation sequent is given by the sum of the individual utilities:

$$u_1^\pi(A) \otimes \cdots \otimes u_n^\pi(A) = sw_u^\pi(A)$$

Slightly abusing the notation, we identify the value $sw_u(A')$ with the value k of the tensor formula u^k. Given two allocations A and A', since we are using LL with (W), we have that $sw_u(A) \leq sw_u(A')$ iff $sw_u(A') \vdash sw_u(A)$. In order to define a strict order, we put $sw_u(A) < sw_u(A')$ iff $sw_u(A') \vdash sw_u(A) \otimes u$.

We can present now the definition of Pareto optimality.

Definition 11 (Pareto optimality)**.** *An allocation A is Pareto optimal iff there is no allocation A' such that $sw(A') \vdash sw(A) \otimes u$ and for all i, $u_i(A) \vdash u_i(A')$.*

6 DEALS

In this section, we define a general language to express deals, then in the next section we will see what it means for an agent to be willing to accept a deal. The language we define will be more general than one would expect, since we consider any kind of formula to be a deal.

We will not put structural constraints on the formula expressing deals; rather, the condition we will put on the feasibility of the negotiation will provide the expected meaning of deals, namely that they transform an allocation A into and allocation A'.

Definition 12. *A deal is any formula of linear logic built over the indexed alphabet $\mathcal{A}^j$.*

So for example a single atom p^j means that p goes to agent j. The meaning of a deal of the form $p^1 \multimap q^3$ is simply the agent 1 loses p and the agent 3 gets q.

Definition 13. *We say that an allocation A' is obtained from A by a* DEAL *iff*

$$A, \text{DEAL} \vdash A'$$

The fact that we use provability to model the passage from a A to A' amounts to assuming that the deals are feasible in the sense that they concern the resources in A. For example, take $p^1 \multimap p^2$; if agent 1 does not own p in A, then such a deal will not be used.

Remark 14. *There are some situations we are excluding. The valuations we are considering are defined on multisets that are represented in our language by tensor formulas. We will not consider here valuations defined on other types of formulas, as options like $a \,\&\, b$ (agent has the choice) or $a \oplus b$ (agent doesn't have the choice): it would require a rather different definition of valuation functions. We leave such extensions to future work.*

We discuss some examples. Deals that simply move a single resource p from one agent to another (1-deals) can be modelled as implications of the form $p^i \multimap p^j$. A *swap deal* [17] between individuals is defined by the following formula $(p^i \multimap p^j) \otimes (q^j \multimap q^i)$, which means that i gives p to j and j gives q to i. For example, let $A = \{p^1, q^2, r^3\}$, we can get $A' = \{p^2, q^1, r^3\}$ by the swap:

$$p^1, q^2, r^3, (p^1 \multimap p^2) \otimes (q^2 \multimap q^1) \vdash p^2 \otimes q^1 \otimes r^3$$

Note that, according to this definition, there might be deals that change nothing, e.g., $p^i \multimap p^i$. Moreover, we can also consider deals that simply provide a resource p to a certain agent i, p^i. In this way, we can for example model, as a form of negotiation, the passage form a partial allocation, in which some goods were not allocated, to a total one:

$$\underbrace{p^1, q^2}_{A}, \underbrace{p^1 \multimap p^2, q^3}_{\text{DEAL}} \vdash \underbrace{p^2 \otimes q^2 \otimes q^3}_{A'}$$

Cluster deals [17], where agents exchange more then one item, can be modelled using tensors: $p^i \otimes q^i \otimes r^i \multimap p^j \otimes q^j \otimes r^j$, meaning that i gives one p, one q and one r to j.

The language of LL allows for expressing deals that entail some forms of choice. Let us call them *optative deals*. So, for example, $(p^1 \multimap p^3) \,\&\, (p^2 \multimap p^3)$ means that 3 would get p from 1 or from 3 (but not from both), or $(p^1 \multimap p^2) \,\&\, (p^1 \multimap p^2)$ means that 1 would give p to 2 or to 3 (but not to both).

Using the distributivity law of LL:

$$A \parr (B \,\&\, C) \dashv\vdash (A \parr B) \,\&\, (A \parr C),$$

we can write optative deals in the following forms. We can express deals like "someone gives p to i" as follows:

$$(p^1 \oplus \cdots \oplus p^n) \multimap p^i$$

Symmetrically, we can express "i gives p to someone":

$$p^i \multimap (p^1 \,\&\, \cdots \,\&\, p^n)$$

In an analogous way, we can consider "i gives something to j" and "i gets something from j".

Taking the language of deals in its full generality, we can also define *transformations* of deals, for example $(p^i \multimap p^j) \multimap (r^j \multimap r^i)$, the intuitive meaning of which is that j would give r to i if the deal $(p^i \multimap p^j)$ has been accepted in the negotiation.

Example 15. *Let A be $\{p^1, r^3, p^1, q^2\}$ and the deals $p^1 \multimap p^2$ and $q^2 \multimap q^3$, meaning that 1 gives on p to 2 and 2 gives one q to 3.*

The following proof shows that $A' = \{p^1, r^3, p^2, q^3\}$ is obtained from A:

$$\dfrac{p^1, r^3 \vdash p^1 \otimes r^3 \qquad \dfrac{p^1, p^1 \multimap p^2 \vdash p^2 \qquad q^2, q^2 \multimap q^3 \vdash q^3}{p^1, q^2, p^1 \multimap p^2, q^2 \multimap q^3 \vdash p^2 \otimes q^3}\otimes L}{p^1, r^3, p^1, q^2, p^1 \multimap p^2, q^2 \multimap q^3 \vdash p^1 \otimes r^3 \otimes p^2 \otimes q^3}\otimes L$$

We can prove that the language of deals is sufficiently powerful to express every transformation of allocations A, A'.

Proposition 16. *Let A and A' be two allocations. Then there exists a formula* DEAL *in the deal language such that*

$$A, \text{DEAL} \vdash A'$$

The proof is obvious in the sense that it is enough to consider the formula $A \multimap A'$ as a deal. We can define a general notion of negotiation as follows.

Definition 17. *A negotiation is a sequent $A, \text{DEAL}_1, \ldots, \text{DEAL}_l \vdash A'$ where $\text{DEAL}_1, \ldots, \text{DEAL}_l$ are accepted deals according to some criterion.*

We can also consider the feasibility of an allocation with respect a given multiset of resources as follows:

$$\mathcal{M}, \text{MAP} \vdash A \tag{2}$$

Here, MAP is the formula defined as in (1). The provability of (2) entails that, given the actual multiset of resources $\mathcal{M}$, A is a feasible way to assign goods.

7 RATIONALITY OF DEALS

In this section, we present some conditions that specify when an agent would accept a deal. Basically, according to the relevant literature [3], we distinguish two cases, one with *side payments* and one without. A *payment function* is a function $p : \mathcal{N} \to \mathbb{Z}$ such that

$$\sum_{i \in \mathcal{N}} p(i) = 0$$

Using side payments, the notion of individual rationality can be defined as follows. A deal is individually rational iff whenever A' is obtained by A by means of that deal, then there exist a payment function p such that for all $i \in \mathcal{N}$:

$$v_i(A') > p(i) + v_i(A)$$

We rephrase the notion of payment function considering formulas in our language as side payments. The requirement that the prices actually paid must sum up to zero is here interpreted as the provability of the sequent containing positive and negative payments. Intuitively, there should be a matching between who pays and who gets payments.

Definition 18. *A side payment is a sequent $X \dashv\vdash Y$, where X and Y are tensors of u, that is provable in LL. We call the formulas on the left negative payments and those on the right positive payments.*

We could also consider more general formulas as side payments. As an example of possible generalisation, we can consider an individual i who would accept to face a loss of three units of her utility for getting one q; it can be modelled using the formula $u^3 \multimap q^i$. However, it is not clear how to define a notion of rationality for side payments consisting of general formulas.

Using payment sequents we can rephrase the notion of individual rationality as follows.

Definition 19. *Given a deal* DEAL *such that A' is obtained by A by means of* DEAL *and a side payment $X \dashv\vdash Y$, we say that* DEAL *is individually rational iff for all i, $u_i(A'_i), X_i \vdash u_i(A_i) \otimes Y_i$ and there exists a j such that: $u_j(A'), X_j \vdash u_j(A_j) \otimes u \otimes Y_j$, where $X_1 \otimes \cdots \otimes X_n \cong X$ and $Y_1 \otimes \cdots \otimes Y_n \cong Y$.*

Note that, since we are working with integers, we do not require all agents to experience a (possibly infinitesimally small) improvement, but rather ask that no agent suffers a loss, and at least one of them gains one full unit u. We can derive the case without side payments, by taking the payment sequent to be $\mathbf{1} \vdash \mathbf{1}$, yielding the following definition of *cooperative rationality* [3]:

Definition 20. *A deal formula* DEAL *such that $A, \text{DEAL} \vdash A'$ is cooperatively rational iff for all i, $u_i(A') \vdash u_i(A)$ and there exists a j such that $u_j(A') \vdash u_j(A) \otimes u$.*

In what follows, w.l.o.g., we will consider payments in which, for each i, (at least one of) X_i or Y_i is the tensor unit $\mathbf{1}$.

Example 21. *Suppose we want to determine whether a deal taking us from allocation A to A' is individually rational. Let $u_1(A') = u^{15}$, $u_2(A') = u^{10}$, $u_3(A') = u^5$ and $u_1(A) = u^2$, $u_2(A) = u^1$, $u_3(A) = u^6$. We can define X_i and Y_i as follows:*

$$\begin{array}{ll} u^{15} \vdash u^2 \otimes u^6 & Y_1 = u^6 \\ u^{10} \vdash u^1 \otimes u^2 & Y_2 = u^2 \\ u^5, u^8 \vdash u^6 & X_3 = u^8 \end{array}$$

We have that positive and negative payments match: $u^8 \dashv\vdash u^6 \otimes u^2$.

We can now state the relationship between individual rationality and social welfare by means of the following theorems. The next result corresponds to [3, Lemma 1], except that we get a more precise characterisation in the context of integer valuations: a deal is individually rational if and only if it increases social welfare by at least one unit.

Theorem 22 (Rational deals and social welfare)**.** *A deal formula* DEAL *with $A, \text{DEAL} \vdash A'$ is individually rational iff $sw_u(A') \vdash sw_u(A) \otimes u$.*

Proof. ($\Rightarrow$) Let DEAL be individually rational. We have that for all i, there are payments such that $u_i(A'), X_i \vdash u(A_i) \otimes Y_i$. Moreover, there is an agent h such that $u_h(A'), X_h \vdash u(A_h) \otimes u \otimes Y_h$. Let $i_1, \ldots i_k$ be the set of agents which gets negative payments (those for which Y_i is $\mathbf{1}$) and $l_{k+1}, \ldots, l_n$ those with positive payments (those for which X_i is $\mathbf{1}$). From sequents $u_{i_j}(A'), X_j \vdash u_{i_j}(A)$, by tensor introduction, we get:

$$u_{i_1}(A'), \ldots, u_{i_k}(A'), X_{i_1}, \ldots, X_{i_k} \vdash u_{i_1}(A) \otimes \cdots \otimes u_{i_k}(A) \tag{3}$$

Now consider the negative payments $u_{l_i}(A') \vdash u_{l_i}(A) \otimes Y_{l_i}$. We can split $u_{l_i}(A')$ in two tensors, say u'_{l_i} and u''_{l_i}, such that $u'_{l_i} \vdash u_{l_i}(A)$ and $u''_{l_i} \vdash Y_{l_i}$. In case h is in this group, then we have that $u'_h \vdash u_h(A) \otimes u$. If h was in the previous group, then his utility has already been considered. So, taking all the $u'_{l_i} \vdash u_{l_i}(A)$, by tensor introduction, we have:

$$\begin{array}{c} u_{i_1}(A'), \ldots, u_{i_k}(A'), u'_{l_{k+1}}, \ldots, u'_{l_n}, X_{i_1}, \ldots X_{i_k} \vdash \\ u_{i_1}(A) \otimes \cdots \otimes u_{i_k}(A) \otimes u_{l_{k+1}}(A) \otimes \cdots \otimes u_{l_n}(A) \end{array} \tag{4}$$

Where the formula on the right hand side amounts to $sw(A) \otimes u$. From all the $u''_{l_i} \vdash X_{l_i}$, we build by introducing tensors:

$$u''_{l_{k+1}}, \dots, u''_{l_n} \vdash Y_{l_{k+1}} \otimes \cdots \otimes Y_{l_n} \quad (5)$$

Since $Y_{l_{k+1}} \otimes \cdots \otimes Y_{l_n} \vdash X_{i_1} \otimes \cdots \otimes X_{i_k}$, we have by cut:

$$u''_{l_{k+1}}, \dots, u''_{l_n} \vdash X_{i_1} \otimes \cdots \otimes X_{i_k} \quad (6)$$

We can finally conclude again by cut on (6) and (4):

$$u_{i_1}(A'), \dots, u_{i_k}(A'), u'_{l_{k+1}}, u''_{l_{k+1}}, \dots, u_{l_n}, u''_{l_{nj}} \vdash sw(A) \otimes u$$

Where $u_{i_1}(A'), \dots, u'_{l_n}, u''_{l_n}$ is $sw(A')$. Thus, $sw(A') \vdash sw(A) \otimes u$

($\Leftarrow$) Let A and A' such that $sw_u(A') \vdash sw(A) \otimes u$, where $A, \text{DEAL} \vdash A'$. We prove that there exist a payment sequent $X \dashv\vdash Y$. We define $Z_i = u^{p(i)}$ where $p(i)$ is defined as follows:

$$p(i) = u_i(A') - u_i(A)$$

Moreover, we chose an individual h and we allocate also $u^{sw(A')-sw(A)}$. We have that if $p(i)$ is positive, then $u_i(A') \vdash u_i(A) \otimes Z_i$ and if $p(i)$ is negative $u_i(A') \otimes Z_i \vdash u_i(A)$. Moreover the individual h will have a strict improvement. The provability of $X \dashv\vdash Y$ follows then from the fact that $p(i)$ sum up to zero. □

In a similar way we can prove a result linking cooperative rationality and Pareto improvements [3].

The following result shows that allocations with maximal utilitarian social welfare can be reached from any (suboptimal) allocation A by means of individually rational deals.

Theorem 23. *Let A^* be an allocation with maximal social welfare. Then for any allocation A with lower social welfare there exists an individually rational deal* DEAL *such that $A, \text{DEAL} \vdash A^*$.*

The proof relies on the fact that there always exists a deal to reach A^* from A, by Proposition 16. Since social welfare improves, by Theorem 22 such a deal is individually rational.

It is interesting to remark that, since we are dealing with integer valuations, if we consider any set of rational deals, each of them must make social welfare increase by at least one unit. Thus, if k is the difference between the maximal social welfare and the social welfare of the initial allocation, then we will always reach an optimal allocation by means of any sequence of at most k individually rational deals.

8 CONCLUSION

We saw how a framework for multilateral negotiation over multisets of goods can be embedded in linear logic. Moreover, we defined a general language to express deals as transformations of allocations, which have adequately been interpreted as proofs in LL. We also showed how to use our framework to exhibit some fundamental results in multiagent resources allocation, and we pointed at some interesting differences with respect to the usual treatment (stemming from the use of integers for valuations).

Future work should include an investigation of the complexity of checking the relevant problems, such as feasibility of an allocation or existence of certain class of sufficient deals. Furthermore, it is interesting to further investigate the notion of side payments we provided, since we would be able consider classes of formulas as payments, which could be interesting for example for modelling agents with various rationality constraints, or even to investigate different notions of rationality .

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