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The Consistency of Majority Rule

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Abstract. We propose an analysis of the impossibility results in judgement aggregation by means of a proof-theoretical approach to collective rationality. In particular, we use linear logic in order to analyse the group inconsistencies and to show possible ways to circumvent them.

1 Introduction

Judgement Aggregation (JA) [4, 5], a recent topic in social choice theory, is concerned with the aggregation of logically connected judgements into a collective rational outcome, by means of a procedure that respects certain fairness desiderata. Recently, JA has been discussed also in AI and multiagent systems. Several results in JA show that it is not possible to aggregate individual judgements, usually expressed in classical propositional logic, by means of procedures that balance fairness and efficiency. For instance, the majority rule faces the so called *discursive dilemmas* [4]: even if individual judgements are rational, the outcome that we obtain by majority may not be. In this paper, we approach discursive dilemmas by using the precise analysis of proofs provided by linear logic (LL) [2]. We will radically depart from a standard assumption in JA, namely, that individual and collective rationality have to be of the same type. [2] By contrast, we will assume that individuals reason classically and we will study which is the notion of rationality that may consistently correspond to group reasoning (wrt majority). In particular, we will show that LL provides a notion of group reasoning that views discursive dilemmas as possible mismatches of the winning coalitions that support logically connected propositions. Section 2 contains the approach of LL to proof-theory. Section 3 contains our analysis of dilemmas. In Section 4, we present our theoretical result. Section 5 concludes.

2 Sequent calculi

LL provides a constructive analysis of proofs by taking into account the actual use of hypotheses of reasoning. In particular, the structural rules of sequent calculus *weakening* and *contraction* are no longer valid in LL, as they would allow us to delete or to add arbitrary copies. By dropping them, the rules that define the connectives are split into two classes: the *additives*, that require the contexts of the sequent to be the same, and the *multiplicatives*, that make copies of the contexts. Accordingly, in LL there are two different types of conjunction, $\otimes$ (tensor) and $\&$ (with), and two types of disjunctions, $\parr$ (parallel) and $\oplus$ (plus). Let $\mathcal{A}$ be a set of atoms, the language of LL is defined as follows

$$\mathcal{L}_{LL} ::= \mathcal{A} \mid \sim L \mid L \otimes L \mid L \parr L \mid L \oplus L \mid L \,\&\, L$$

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[2] The discussion of LL for JA points at a generalisation of the approach in [1], because we deal with non-monotonic consequence relations.

The sequent calculus is presented in the following table. We shall assume that (P) always holds. If we assume (W) and (C), than the two rules for the two conjunction coincide. In that case, $\otimes$ and $\&$ collapse and the meaning of the conjunction is the classical one. The same holds for disjunctions. We shall use the usual notation, $a \wedge b$ and $a \vee b$, when we assume that the structural rules hold and we denote $\mathcal{L}_{CL}$ the language of classical logic.

$$\frac{}{A \vdash A}\,\text{ax} \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{cut}$$

Negation

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \sim A \vdash \Delta}\,L\sim \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \sim A, \Delta}\,R\sim$$

Multiplicatives

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\,\otimes\text{L} \qquad \otimes\text{R}\,\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'}\,\parr\text{L} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\,\parr\text{R}$$

Additives

$$\frac{\Gamma, A_i \vdash \Delta}{\Gamma, A_0 \& A_1 \vdash \Delta}\,\&\text{L} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta}\,\&\text{R}$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}\,\oplus\text{L} \qquad \frac{\Gamma \vdash A_i, \Delta}{\Gamma \vdash A_0 \oplus A_1, \Delta}\,\oplus\text{R}$$

Structural Rules (also on the right)

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\,\text{P} \qquad \frac{\Gamma, A, A, \vdash \Delta}{\Gamma, A \vdash \Delta}\,\text{C} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,\text{W}$$

The idea of this work is to model group reasoning by using the linear logic awareness of contexts and inferences. We shall view coalitions of agents that support formulas as contexts in the sequent calculus. For example, if the group accepts a conjunction of two sentences, this might have two interpretations: there exists a single coalition Γ such that $\Gamma \vdash a$ and $\Gamma \vdash b$, therefore $\Gamma \vdash a \,\&\, b$; or there are two different coalitions such that $\Gamma \vdash a$ and $\Delta \vdash b$, therefore $\Gamma, \Delta \vdash a \otimes b$.

3 The model

Let N be a (finite) set of agents and $\mathcal{X}$ an *agenda*, namely, a (finite) set of propositions in the language $\mathcal{L}_L$ of a given logic L that is closed under complements, i.e. (non-double) negations. A *judgement set* J is a subset of $\mathcal{X}$ such that J is (wrt L) *consistent* ($J \nvdash_L \emptyset$), *complete* (for all $\phi \in \mathcal{X}$, $\phi \in J$ or $\sim \phi \in J$) and *deductive closed* (if $J \vdash_L \phi$ and $\phi \in \mathcal{X}$, $\phi \in J$). Let $L(\mathcal{X})$ the set of all judgement sets on $\mathcal{X}$ wrt L. A *profile* of judgement sets $\mathbf{J}$ is a vector $(J_1, \ldots, J_n)$. We assume that individuals reason in CL (just like in standard JA). Different logics may model group reasoning. For example, group reasoning in CL is treated in standard JA. We focus on the case in which group reasoning is modelled by LL. Thus, we need to adapt the notion of aggregator, by adding a translation function from CL into LL. Given an agenda $\mathcal{X} \subset \mathcal{L}_{CL}$, the agenda $\mathcal{X}' \subset \mathcal{L}_{LL}$ is defined by the following *additive translation*: if $\phi \in \mathcal{X}$, then $add(\phi)$ (replace $\wedge$ with $\&$ and $\vee$ with $\oplus$) is in $\mathcal{X}'$. An *aggregator* is then a function $F : CL(\mathcal{X})^n \to LL(\mathcal{C})$ such that F is the composition of a standard aggregator $F' : CL(\mathcal{X})^n \to \mathcal{P}(\mathcal{X})$ and a translation function $t : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X}')$, such that $t(J) = \{add(\phi) \mid \phi \in J\}$.[3] For example, the majority rule is $M(\mathbf{J}) = t(\{\phi \in \mathcal{X} \mid |N_\phi| > n/2\})$ with $N_\phi = \{i \mid \phi \in J_i\}$. N_ϕ is a *winning coalition* W_ϕ if $\phi \in M(\mathbf{J})$. We model group reasoning as follows. We assume non-logical axioms $W_\phi \vdash \phi$ for any $\phi \in F(\mathbf{J})$. Intuitively, the group reasons from accepted formulas keeping track of their winning coalitions.

Definition 1 (Group reasoning) *We say that the group infers a formula $\phi \in \mathcal{L}_L$ according to L iff, for some $W_1, ..., W_m$, there is a proof in L from some of the axioms $W_1 \vdash_L \phi_1, \ldots, W_m \vdash_L \phi_m$ to $W_1, ..., W_m \vdash_L \phi$.*

Note that the group is inconsistent iff, for some $W_1, ..., W_m$, the sequent $W_1, \ldots, W_m \vdash_L \emptyset$ is derivable in L.

3.1 An analysis of discursive dilemmas

Consider the following example of discursive dilemma on the agenda $\{a, b, a \wedge b, \sim a, \sim b, \sim (a \wedge b)\}$.

	a	$a \wedge b$	b	$\sim a$	$\sim (a \wedge b)$	$\sim b$
i_1	1	1	1	0	0	0
i_2	1	0	0	0	1	1
i_3	0	0	1	1	1	0
maj.	1	0	1	0	1	0

Each agent has a consistent set, however, by majority, the collective set $\{a, b, \sim (a \wedge b)\}$ is not. We can infer the contradiction in the collective by reasoning in CL as follows.

$$\dfrac{\dfrac{i_1, i_2 \vdash a}{i_1, i_2, i_3 \vdash a}\,\mathrm{W} \qquad \dfrac{i_1, i_3 \vdash b}{i_1, i_2, i_3 \vdash b}\,\mathrm{W}}{i_1, i_2, i_3 \vdash a \wedge b}\,R\wedge$$

We start with non-logical axioms $i_1, i_2 \vdash a$ and $i_1, i_3 \vdash b$. By weakening, we introduce the conjunction of a and b by using the same coalition. Moreover, the group can infer $\sim (a \wedge b)$ as we have the axiom: $i_2, i_3 \vdash \sim (a \wedge b)$. Therefore, the group is inconsistent wrt CL, as we can prove $a \wedge b$ and $\sim (a \wedge b)$ by using the W_i. This entails, by (cut), that we can prove $\emptyset$ from some W_i.

If we drop W and C, the contradiction is no longer derivable. If the group reasons in LL, the non-logical axioms are: $i_1, i_2 \vdash a$, $i_1, i_3 \vdash b$ and $i_2, i_3 \vdash \sim (a \,\&\, b)$. The only way the group can infer $a \otimes b$ is by using two different coalitions:

$$\dfrac{\{i_1, i_2\} \vdash a \qquad \{i_1, i_3\} \vdash b}{\{i_1, i_2\}, \{i_1, i_3\} \vdash a \otimes b}\,R\otimes$$

However, $a \otimes b$ and $\sim (a \,\&\, b)$ are *not* inconsistent in LL, because $a \otimes b, \sim (a \,\&\, b) \nvdash_{LL} \emptyset$. LL provides then a reasoning method that keeps track of the fact that there is no winning coalition for $a \wedge b$, while there are winning coalitions for a and b. Accordingly, we cannot infer $a \& b$ from any W_i, since there is no single coalition that supports both a and b.

[3] The translation reflects our view: Multiplicatives combine coalitions, whereas additives refer to a same coalition.

4 Consistency wrt group reasoning in LL

According to results in JA [5], the majority rule leads to inconsistency iff the agenda contains a minimally inconsistent set Y such that $|Y| \geq 3$ (e.g. $\{a, b, \sim (a \wedge b)\}$). Moreover, if $Y \subset M(\mathbf{J})$, there must be at least three different winning coalitions supporting the formulas in Y. We prove that majority is always consistent wrt LL, provided our additive translation. The key property is the following: $(\mathcal{F}_2)$ if we restrict to additive linear logic (ALL) ($\&$ and $\oplus$), every provable sequent contains at most two formulas (e.g. $A \vdash B$) [3]. [4]

Theorem 1 *For every $\mathcal{X} \in \mathcal{L}_{CL}$, if every J_i is consistent wrt CL, and n is odd, then the majority rule is always consistent wrt group reasoning in LL.*

Proof. If $M(\mathbf{J})$ is consistent wrt CL, then it is consistent wrt LL: if $M(\mathbf{J}) \nvdash_{CL} \emptyset$, then $t(M(\mathbf{J})) \nvdash_{LL} \emptyset$ (as in LL we use less rules). Suppose there is a minimally inconsistent $Y \subset \mathcal{X}$ s.t. $|Y| \geq 3$. Let $\mathbf{J}$ be a profile s.t. $Y \subseteq M(\mathbf{J})$. We show that the group is consistent wrt LL on $t(Y)$. For any $\phi_i \in t(Y)$, we have axioms $W_i \vdash \phi_i$. All the formulas in $t(Y)$ are additive, thus, by property $(\mathcal{F}_2)$, the only ways to prove $\emptyset$ from the formulas in $t(Y)$ are: 1) to prove $A \vdash \emptyset$, with $A = \&_i \phi_i$, $\phi_i \in Y$, and 2) to prove $B, C \vdash \emptyset$, where $B = \&_i \phi_i$ and $C = \&_j \phi_j$, with $\phi_i \neq \phi_j \in t(Y)$. The only way to prove $A = \&_i \phi_i$ from some winning coalitions W_i is by means of a single W, s.t. $W \vdash \phi_i$ for every $\phi_i \in t(Y)$, against the consistency of each J_i. The only way to prove B and C (i.e. $B \otimes C$) from some W_i is to have two winning coalitions W and W' s.t. W supports all ϕ_i and W' supports all ϕ_j. Again, this is against the consistency of each J_i, as there must be an i supporting the full Y.□

5 Conclusion

We have shown that majority is consistent wrt a notion of group reasoning defined in LL. A reasoning method based on LL has several independent applications as reasoning on bounded resources and as a logic of computation [2]. Here, we have seen that LL provides a notion of group rationality that views discursive dilemmas as mismatches of winning coalitions wrt majority rule. The significance of applying proof-theoretical methods to JA is that they link possibility results to a fine-grained analysis of reasoning and, by inspecting logical rules, we may draw a new map of possibility/impossibility results. A similar treatment can be developed also for preference aggregation and can be generalised to classes of aggregators. Future work shall investigate this aspects.

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[4] If we inspect the additive rules, we see that they cannot add any new proposition. Note that $(\mathcal{F}_2)$ entails that in ALL there are no minimal inconsistent sets of size greater than 3. Thus majority is safe for any ALL agenda. This result is of an independent interest as it provides a new possibility result that links language restrictions to reasoning methods.