Non-normal modalities in variants of Linear Logic*

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Abstract

This note presents modal versions of resource-conscious logics. We concentrate on extensions of variants of Linear Logic with one minimal non-normal modality. In earlier work, where we investigated agency in multi-agent systems, we have shown that the results scale up to logics with multiple non-minimal modalities. Here, we start with the language of propositional intuitionistic Linear Logic without the additive disjunction, to which we add a modality. We provide an interpretation of this language on a class of Kripke resource models extended with a neighbourhood function: modal Kripke resource models. We propose a Hilbert-style axiomatization and a Gentzen-style sequent calculus. We show that the proof theories are sound and complete with respect to the class of modal Kripke resource models. We show that the sequent calculus allows cut elimination and that proof-search is in PSPACE. We then show how to extend the results when non-commutative connectives are added to the language. Finally, we put the logical framework to use by instantiating it as logics of agency. In particular, we propose a logic to reason about the resource-sensitive use of artefacts and illustrate it with a variety of examples.

1 Introduction

Logics for resources and modalities each got their share of the attention and have also already been studied together. Extensions of intuitionistic linear logic with modalities have been investigated for example in (D'Agostino et al., 1997; Kamide, 2006; Marion and Sadrzadeh, 2004). Moreover, modal versions of logics for resources that are related to Linear Logic have been provided in (Courtault and Galmiche, 2013; Pym and Tofts, 2006; Pym et al., 2004).

Modalities in substructural logics are thus not new, although with one important detail: to our knowledge, modalities in sub-structural logics have always

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been restricted to *normal* modalities. Modalities that are not normal have been confined to the realm of classical logic.

Normal modalities are the modalities within a logic that is at least as strong as the standard modal logic K. *Non-normal modalities* on the other hand are the modalities within a logic strictly weaker than the standard modal logic K. They cannot be evaluated over a Kripke semantics but one typical semantics, neighbourhood models, rely on possible worlds. They were introduced independently by Scott and Montague. Early results were offered by Segerberg. Chellas built upon and gave a textbook presentation in (Chellas, 1980).

The significance of non-normal modal logics and their semantics in modern developments in logics of agents has been emphasised before (Arló-Costa and Pacuit, 2006). Indeed many logics of agents are non-normal and neighbourhood semantics allows for defining the modalities that are required to model a number of application domains: logics of coalitional power (Pauly, 2002), epistemic logics without omniscience (Lismont and Mongin, 1994; Vardi, 1986), logics of agency (Governatori and Rotolo, 2005), etc.

Let us briefly present some features of reasoning about agency specifically. Modalities of agency aimed at modelling the result of an action have been largely studied in the literature in practical philosophy and in multi-agent systems (Belnap et al., 2001; Governatori and Rotolo, 2005; Kanger and Kanger, 1966; Pörn, 1977; Troquard, 2014). Logics of agency assume a set of agents, and to each agent i associate a modality E_i. We read $\mathsf{E}_i\varphi$ as "agent i brings about φ". With the notable exception of Chellas' logic (Chellas, 1969), it is generally accepted in logics of agency that no agent ever brings about a tautology. This is because when i brings about φ, it is intended that φ might have not be the case if it were not for i's very agency. Classically, it corresponds to the axiom $\neg\mathsf{E}_i\top$. This principle is enough for the logic of the modality E_i to be non-normal as it is inconsistent with the necessitation rule. A normal modal logic for agency would also dictate that if i does φ then she also does $\varphi \vee \psi$. This is refuted in general on the same grounds as Ross' paradox. For instance, purposefully doing that I am rich should not imply purposefully doing that I am rich or unhealthy.

In classical logic, $\mathsf{E}_i\varphi$ allows one to capture that agent i brings about *the state of affairs* φ. Moving from states of affairs to resources, it is then interesting to lift these modalities from classical logic to resource-conscious logic. In doing so, one can capture with $\mathsf{E}_i A$ that agent i brings about *the resource* A.

To make a start with this research program, we will combine intuitionistic fragments of Linear Logic with non-normal modalities. Linear Logic (Girard, 1987) is a resource-conscious logic that allows for modelling the constructive content of deductions in logic. An *intuitionistic* version of Linear Logic, as the one we will work with, has a number of desirable features. One that is simple and yet greatly appreciable is that as we shall see, in intuitionistic sequent calculus every sequent has a single "output" formula. This feature favours the modelling of input-output processes. It will prove particularly adequate for our application to agency and artefacts as it provides a simple mechanism for the compositionality of individual artefacts' functions into complex input-output processes.

The resource-sensitive nature of Linear Logic can be viewed as the absence of structural rules in the sequent calculus. Linear Logic rejects the global validity of weakening (W), that amounts to a monotonicity of the entailment, and contraction (C), that is responsible for arbitrary duplications of formulas, e.g., $A \to A \wedge A$ is a tautology in classical logic, and so is $A \wedge A \to A$.

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\,(\mathrm{W}) \qquad \frac{\Gamma, B, B \vdash A}{\Gamma, B \vdash A}\,(\mathrm{C}) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, B, A \vdash C}\,(\mathrm{E})$$

Accordingly, the linear implication $\multimap$ encodes resource-sensitive deductions, for example from A and $A \multimap B$ we can infer B, by *modus ponens*, but we are not allowed to conclude B from A, A, and $A \multimap B$. Exchange (E) still holds. (Although we will later restrict it.) Hence, contexts of formulas Γ in sequent calculus are considered multisets. By dropping weakening and contraction, we are led to define two non-equivalent conjunctions with different behaviours: the multiplicative conjunction $\otimes$ (tensor) and the additive conjunction & (with). The intuitive meaning of $\otimes$ is that an action of type $A \otimes B$ can be performed by summing the resources that are relevant to perform A and to perform B. The unit $\mathbf{1}$ is the neutral element for $\otimes$ and can represent model a null action. A consequence of the lack of weakening is that $A \otimes B$ no longer implies A, namely the resources that are relevant to perform $A \otimes B$ may not be relevant to perform just A. The absence of contraction means that $A \multimap A \otimes A$ is no longer valid. The additive conjunction $A \,\&\, B$ expresses an option, the choice to perform A or B. Accordingly $A \,\&\, B \multimap A$ holds in Linear Logic, the resources that enable the choice between A and B are relevant also to make A or to make B. The linear implication $A \multimap B$ expresses a form of causality, for example "If I strike a match, I can light the room" the action of striking a match is consumed, in the sense that it is no longer available after the room is lighted. Linear implication and multiplicative conjunction interact naturally so that $(A \otimes (A \multimap B)) \multimap B$ is valid.

Linear logic operators have been applied to a number of topics in knowledge representation and multiagent systems such as planning (Kanovich and Vauzeilles, 2001), preference representation and resource allocation (Harland and Winikoff, 2002; Porello and Endriss, 2010), social choice (Porello, 2013), and actions modelling (Borgo et al., 2014).

These propositional operators are very useful when talking about resources and agency. They allow one to capture the following notions:

- Bringing about both A and B together: $\mathsf{E}_i(A \otimes B)$. In such a way that $\mathsf{E}_i(A \otimes B)$ implies $A \otimes B$, but does not imply A alone.
- Bringing about an option between A and B: $\mathsf{E}_i(A \,\&\, B)$. In such a way that $\mathsf{E}_i(A \,\&\, B)$ implies A and implies B, but does not imply $A \otimes B$.
- Bringing about the transformation of the resource A into the resource B: $\mathsf{E}_i(A \multimap B)$. In such as way that $A \otimes \mathsf{E}_i(A \multimap B)$ implies B.

The only structural rule that holds in Linear Logic is exchange (E), that is responsible for the commutativity of the logical operators and amounts to forgetting sequentiality of information, e.g., $A \wedge B \to B \wedge A$. We will see in this paper how to deal with ordered information. We will still admit exchange for the two conjunctions & and $\otimes$ (commutative conjunctions), but we will introduce a non-commutative counterpart $\odot$ to the multiplicative $\otimes$. The formula $A \odot B$ is not equivalent to $B \odot A$. Since $\odot$ is non-commutative, we can have two *order-sensitive* linear implication (noted in (Lambek, 1958) $\setminus$ and $/$).

For agency in multi-agent systems, resources must often become available at key points in a series of transformations. The order of resource production and transformation becomes quickly relevant. It is then interesting to talk about:

- Bringing about the resource A first, then B: $\mathsf{E}_i(A \odot B)$. In such a way that $\mathsf{E}_i(A \odot B)$ is not equivalent to $\mathsf{E}_i(B \odot A)$.
- Bringing about the order-sensitive transformation of the resource from A into B: $\mathsf{E}_i(A \setminus B)$. In such a way that $A \odot \mathsf{E}_i(A \setminus B)$ implies B, but $\mathsf{E}_i(A \setminus B) \odot A$ does not.

These considerations motivate us to investigate the theoretical underpinnings of modal versions of resource-conscious logics with the listed propositional operators.

We will first concentrate on extensions with <u>one</u> <u>minimal</u> <u>non-normal</u> modality. In a second part, where we address modalities of agency, we will exploit our results that will naturally scale up to logics with <u>multiple</u> <u>non-minimal</u> (but still non-normal) modalities.

Outline. In Section 2, we present the Kripke resource models which already exist in the literature. The semantics of all the languages studied in this paper will be adequate extensions of Kripke resource models. We first enrich the Kripke resource models with neighbourhood functions to capture non-normal modalities. We obtain what we simply coin modal Kripke resource models. We define and study a minimal non-normal modal logic, MILL. We introduce a Hilbert system in Section 3 and a sequent calculus in Section 4; Both are shown sound and complete. Moreover, we show that the sequent calculus allows cut elimination that provides a normal form for proofs. Proof search is proved to be in PSPACE.

We extend our framework in Section 5 to account for partially commutative Linear Logic that allows for integrating commutative and non-commutative operators.

Then in Section 6 we instantiate the minimal modal logic with a resource-sensitive version of the logics of bringing-it-about: RSBIAT. Again, sound and complete Hilbert system and sequent calculus are provided. Proof-search in RSBIAT is PSPACE-easy. We also show how RSBIAT can be extended with the non-commutative language and thus represent sequentiality of actions. In Section 7, we motivate and discuss a number of applications of our system to represent and reason about artefacts.

2 MILL and modal Kripke resource models

We introduce the most basic language studied in this paper. The language $\mathcal{L}_{\mathsf{MILL}}$ is given by the BNF:

$$A ::= \mathbf{1} \mid p \mid A \otimes A \mid A \,\&\, A \mid A \multimap A \mid \Box A$$

where $p \in Atom$. It is a modal version of what corresponds to the language of propositional intuitionistic Linear Logic, but without the additive disjunction and the additive units. This propositional part can also be seen as the fragment of BI (O'Hearn and Pym, 1999) without additive disjunction and implication.

Let us first concentrate on the propositional part for which a semantics already exists in the literature. We call the logic ILL and $\mathcal{L}_{\mathsf{ILL}}$ its language. A Kripke-like class of models for ILL is basically due to Urquhart (Urquhart, 1972). A *Kripke resource frame* is a structure $\mathcal{M} = (M, e, \circ, \geq)$, where $(M, e, \circ)$ is a commutative monoid with neutral element e, and $\geq$ is a pre-order on M. The frame has to satisfy the condition of *bifunctoriality*: if $m \geq n$, and $m' \geq n'$, then $m \circ m' \geq n \circ n'$. To obtain a *Kripke resource model*, a valuation on atoms $V : Atom \to \mathcal{P}(M)$ is added. It has to satisfy the *heredity* condition: if $m \in V(p)$ and $n \geq m$ then $n \in V(p)$.

The truth conditions of $\mathcal{L}_{\mathsf{ILL}}$ in the Kripke resource model $\mathcal{M} = (M, e, \circ, \geq, V)$ of the formulas of the propositional part are the following:

$m \models_{\mathcal{M}} p$ iff $m \in V(p)$.

$m \models_{\mathcal{M}} \mathbf{1}$ iff $m \geq e$.

$m \models_{\mathcal{M}} A \otimes B$ iff there exist m_1 and m_2 such that $m \geq m_1 \circ m_2$ and $m_1 \models_{\mathcal{M}} A$ and $m_2 \models_{\mathcal{M}} B$.

$m \models_{\mathcal{M}} A \,\&\, B$ iff $m \models_{\mathcal{M}} A$ and $m \models_{\mathcal{M}} B$.

$m \models_{\mathcal{M}} A \multimap B$ iff for all $n \in M$, if $n \models_{\mathcal{M}} A$, then $n \circ m \models_{\mathcal{M}} B$.

Observe that heredity can be shown to extend naturally to every formula, in the sense that:

Proposition 1. *For every formula $A \in \mathcal{L}_{\mathsf{ILL}}$, if $m \models A$ and $m' \geq m$, then $m' \models A$.*

Notations. An intuitionistic negation can be added to the language. We simply choose a designated atom $\perp \in Atom$ and decide to conventionally interpret $\perp$ as indicating a contradiction. Negation is then defined by means of implication as $\sim A \equiv A \multimap \perp$ (Kanovich et al., 2006): the occurrence of A yields the contradiction. There will be no specific rule for negation.

Given a multiset of formulas, it will be useful to combine them into a unique formula. We adopt the following notation: $\emptyset^* = \mathbf{1}$, and $\Delta^* = A_1 \otimes \cdots \otimes A_k$ when $\Delta = \{A_1, \ldots, A_k\}$.

Denote $||A||^{\mathcal{M}}$ the extension of A in $\mathcal{M}$, i.e. the set of worlds of $\mathcal{M}$ in which A holds. A formula A is *true* in a model $\mathcal{M}$ if $e \models_{\mathcal{M}} A$.[1] A formula A is *valid* in Kripke resource frames, noted $\models A$, iff it is true in every model.

Modal Kripke resource models. Now, to give a meaning to the modality, we define a neighbourhood semantics on top of the Kripke resource frame. A neighbourhood function is a mapping $N : M \to \mathcal{P}(\mathcal{P}(M))$ that associates a world m with a set of sets of worlds. (See (Chellas, 1980).) We define:

$m \models \Box A$ iff $||A|| \in N(m)$

This is not enough, though. It is possible that $m \models \Box A$, yet $m' \not\models \Box A$ for some $m' \geq m$. That is, Proposition 1 does not hold with the simple extension of $\models$ for $\mathcal{L}_{\mathsf{MILL}}$. (One disastrous consequence is that the resulting logic does not satisfy the *modus ponens* or the cut rule.) We could define the clause concerning the modality alternatively as: $m \models \Box A$ iff there is $n \in M$, such that $m \geq n$ and $||A|| \in N(n)$. However, this is bothersome because this is not how a non-normal modality is traditionally defined (Chellas, 1980).

Instead, we will require our neighbourhood function to satisfy the condition that if some set $X \subseteq M$ is in the neighbourhood of a world, then X is also in the neighbourhood of all "greater" worlds.[2] Formally, our modal Linear Logic is evaluated over the following models:

Definition 1. *A* modal Kripke resource model *is a structure* $\mathcal{M} = (M, e, \circ, \geq, N, V)$ *such that:*

- $(M, e, \circ, \geq)$ *is a Kripke resource frame;*
- N *is a neighbourhood function such that:*

$$\text{if } X \in N(m) \text{ and } n \geq m \text{ then } X \in N(n) \qquad (1)$$

It is readily checked that Proposition 1 is true as well for $\mathcal{L}_{\mathsf{MILL}}$ over modal Kripke resource models for modal formulas. We thus have:

Proposition 2. *For every formula* $A \in \mathcal{L}_{\mathsf{MILL}}$*, if* $m \models A$ *and* $m' \geq m$*, then* $m' \models A$*.*

3 Hilbert system for MILL and soundness

This part extends the Hilbert system for ILL from (Troelstra, 1992) and (Avron, 1988). We define the Hilbert-style calculus H-MILL for MILL by defining the following notion of deduction.

[1] When no confusion can arise we will write $||A||$ instead of $||A||^{\mathcal{M}}$, and $m \models A$ instead of $m \models_{\mathcal{M}} A$.

[2] An analogous yet less transparent condition was used in (D'Agostino et al., 1997) for a normal modality.

$A \multimap A$
$(A \multimap B) \multimap ((B \multimap C) \multimap (A \multimap C))$
$(A \multimap (B \multimap C)) \multimap (B \multimap (A \multimap C))$
$A \multimap (B \multimap A \otimes B)$
$(A \multimap (B \multimap C)) \multimap (A \otimes B \multimap C)$
$\mathbf{1}$
$\mathbf{1} \multimap (A \multimap A)$
$(A \,\&\, B) \multimap A$
$(A \,\&\, B) \multimap B$
$((A \multimap B) \,\&\, (A \multimap C)) \multimap (A \multimap B \,\&\, C)$

Table 1: Axiom schemata in H-MILL

Definition 2 (Deduction in H-MILL)**.** *A* deduction tree *in* H-MILL $\mathcal{D}$ *is inductively constructed as follows. (i) The leaves of the tree are assumptions* $A \vdash_{\mathsf{H}} A$*, for* $A \in \mathcal{L}_{\mathsf{MILL}}$*, or* $\vdash_{\mathsf{H}} B$ *where* B *is an axiom in Table 1 (base cases).*

(ii) We denote by $\Gamma \overset{\mathcal{D}}{\vdash_{\mathsf{H}}} A$ *a deduction tree with conclusion* $\Gamma \vdash_{\mathsf{H}} A$*. If* $\mathcal{D}$ *and* $\mathcal{D}'$ *are deduction trees, then the following are deduction trees (inductive steps).*

$$\frac{\Gamma \overset{\mathcal{D}}{\vdash_{\mathsf{H}}} A \qquad \Gamma' \overset{\mathcal{D}'}{\vdash_{\mathsf{H}}} A \multimap B}{\Gamma, \Gamma' \vdash_{\mathsf{H}} B} \multimap\text{-}rule \qquad \frac{\Gamma \overset{\mathcal{D}}{\vdash_{\mathsf{H}}} A \qquad \Gamma \overset{\mathcal{D}'}{\vdash_{\mathsf{H}}} B}{\Gamma \vdash_{\mathsf{H}} A \,\&\, B} \,\&\text{-}rule$$

$$\frac{\overset{\mathcal{D}}{\vdash_{\mathsf{H}}} A \multimap B \qquad \overset{\mathcal{D}'}{\vdash_{\mathsf{H}}} B \multimap A}{\vdash_{\mathsf{H}} \Box A \multimap \Box B} \,\Box(re)$$

We sometimes refer to the $\multimap$-rule as *modus ponens*. We say that A is deducible from Γ in H-MILL and we write $\Gamma \vdash_{\mathsf{H\text{-}MILL}} A$ iff there exists a deduction tree in H-MILL with conclusion $\Gamma \vdash_{\mathsf{H}} A$. The deduction without assumptions, i.e. $\vdash_{\mathsf{H\text{-}MILL}} A$, is just a special case of the above definition.

In general when defining Hilbert systems for linear logics, we need to be careful in the definition of derivation from assumptions: since Γ is a multiset, we need to handle occurrences of hypothesis when applying for instance modus ponens. From Definition 2, every formula in a derivation, except for the conclusion, is used exactly once as premise of modus ponens (Avron, 1988). With respect to this notion of derivation, the deduction theorem holds.

Theorem 3 (Deduction theorem for H-MILL)**.** *If* $\Gamma, A \vdash_{\mathsf{H\text{-}MILL}} B$ *then* $\Gamma \vdash_{\mathsf{H\text{-}MILL}} A \multimap B$*.*

Proof. The deduction theorem holds for the propositional fragment ILL, see (Troelstra, 1992, pp. 66-68). The only case to consider is the rule $\Box$(re). However this trivially holds because the contexts of the sequents in the rule are empty. □

Remark. *We defined the rule $\Box$(re) in that particular way because of the deduction theorem. With the above formulation, the deduction theorem is preserved in* MILL*. Moreover, by our definition of deduction tree, this version entails the rule $\Box$(re'): from two deductions trees with conclusion $A \vdash_{\mathsf{H}} B$ and $B \vdash_{\mathsf{H}} A$, we can build a deduction tree with conclusion $\Box A \vdash_{\mathsf{H}} \Box B$.*

We claim that $\Box$(re) implies $\Box$(re'). Assume the premises of $\Box$(re'): there are two deduction trees with conclusions $A \vdash_{\mathsf{H}} B$ and $B \vdash_{\mathsf{H}} A$. In virtue of Theorem 3 and the definition of $\vdash_{\mathsf{H\text{-}MILL}}$, we know there are two trees with conclusions $\vdash_{\mathsf{H}} A \multimap B$ and $\vdash_{H} B \multimap A$. Thus by $\Box$(re), we can build a deduction tree with conclusion $\vdash_{\mathsf{H}} \Box A \multimap \Box B$. Together with $\Box A \vdash_{\mathsf{H}} \Box A$ (a leaf), the $\multimap$-rule gives us a deduction tree with conclusion $\Box A \vdash_{\mathsf{H}} \Box B$.

With the version $\Box$(re'), the deduction theorem fails, as we do not have any axiom that talks about the modality in this case.

We can prove the soundness of H-MILL wrt. our semantics.

Theorem 4 (Soundness of H-MILL)**.** *If $\Gamma \vdash_{\mathsf{H\text{-}MILL}} A$ then, for every model, $\Gamma \models A$ (namely, $e \models (\Gamma)^* \multimap A$).*

Proof. We only give the arguments of the proof of soundness for two representative cases.

Soundness of $\multimap$-rule. We show now that $\multimap$-rule preserves validity. Namely, we prove, by induction on the length of the derivation tree that if (1) $e \models \Gamma \multimap A$ and (2) $e \models \Gamma' \multimap (A \multimap B)$, then $e \models \Gamma \otimes \Gamma' \multimap B$. The first assumption entails that for all x, if $x \models \Gamma$, then $x \models A$. The second assumption entails that for all y, if $y \models \Gamma'$, then $y \models A \multimap B$. Thus, for all t, if $t \models A$, then $y \circ t \models B$.

Let $z \models \Gamma \otimes \Gamma'$, thus there exist z_1 and z_2 such that $z_1 \models \Gamma$, $z_2 \models \Gamma'$ and $z \geq z_1 \circ z_2$. By (1), we have that $z_1 \models A$ and, by (2), $z_2 \models A \multimap B$, thus $z_1 \circ z_2 \models B$, and by Proposition 2 we have that $z \models B$.[3]

Soundness of $\Box$(re). We show that $\Box$(re) preserves validity, namely, if $e \models A \multimap B$ and $e \models B \multimap A$, then $e \models \Box A \multimap \Box B$. Our assumptions imply that, for all x, if $x \models A$, then $x \models B$, and if $x \models B$ then $x \models A$. Thus, $||A|| = ||B||$. We need to show that for all x, if $x \models \Box A$, then $x \models \Box B$. By definition, $x \models \Box A$ iff $||A|| \in N(x)$. Thus, since $||A|| = ||B||$, we have that $||B|| \in N(x)$, that means $x \models \Box B$. □

4 Sequent calculus MILL and completeness

In this section, we introduce the sequent calculus for our logic. A *sequent* is a statement $\Gamma \vdash A$ where Γ is a finite multiset of occurrences of formulas of MILL and A is a formula. The fact that we allow for a single formula in the conclusions of the sequent corresponds to the fact that we are working with the intuitionistic version of the calculus (Girard, 1987).

[3] Remember that Proposition 2 holds for the modal language $\mathcal{L}_{\mathsf{MILL}}$ because of Condition (1) on modal Kripke resource models.

$$\frac{}{A \vdash A}\,\text{ax} \qquad \frac{\Gamma, A \vdash C \qquad \Gamma' \vdash A}{\Gamma, \Gamma' \vdash C}\,\text{cut}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,\otimes\text{L} \qquad \frac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}\,\otimes\text{R}$$

$$\frac{\Gamma \vdash A \qquad \Gamma', B \vdash C}{\Gamma', \Gamma, A \multimap B \vdash C}\,\multimap\text{L} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,\multimap\text{R}$$

$$\frac{\Gamma, A, \Gamma' \vdash C}{\Gamma, A \,\&\, B, \Gamma' \vdash C}\,\&\text{L} \qquad \frac{\Gamma, B, \Gamma' \vdash C}{\Gamma, A \,\&\, B, \Gamma' \vdash C}\,\&\text{L}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\,\&\text{R} \qquad \frac{\Gamma \vdash C}{\Gamma, \mathbf{1} \vdash C}\,\mathbf{1}\text{L} \qquad \frac{}{\vdash \mathbf{1}}\,\mathbf{1}\text{R}$$

$$\frac{A \vdash B \qquad B \vdash A}{\Box A \vdash \Box B}\,\Box(\text{re})$$

Table 2: Sequent calculus MILL

Since in a sequent $\Gamma \vdash A$ we identify Γ to a multiset of formulas, the exchange rule—the reshuffling of Γ—is implicit.

A sequent $\Gamma \vdash A$ where $\Gamma = A_1, \ldots, A_n$ is *valid* in a modal Kripke resource frame iff the formula $A_1 \otimes \ldots \otimes A_n \multimap A$ is valid, namely $\models \Gamma^* \multimap A$.

We obtain the sequent calculus for our minimal modal logic MILL by extending the language of ILL with modal formulas and by adding a new rule. Saturating the notation, we label the sequent calculus rule like the rule for equivalents in the Hilbert system: $\Box$(re). The calculus is shown in Table 2.

To establish a link with the previous section, we first show that provability in sequent calculus is equivalent to provability in the Hilbert system.

Theorem 5. *It holds that $\Gamma \vdash_{\text{H-MILL}} A$ iff the sequent $\Gamma \vdash A$ is derivable in the sequent calculus for* MILL.

Proof. (Sketch) The propositional cases are proved in (Avron, 1988), we only need to extend it for the case of the modal rules. This is done by induction on the length of the proof. For instance, in one direction, assume that a derivation $\mathcal{D}$ of $\vdash_H \Box A \multimap \Box B$ is obtained by $\mathcal{D}'$ of $\vdash_H A \multimap B$ and $\mathcal{D}''$ of $\vdash_H B \multimap A$. By definition of deduction and of $\vdash_{\text{H-MILL}}$, we have that $A \vdash_{\text{H-MILL}} B$ and $B \vdash_{\text{H-MILL}} A$. By induction hypothesis, we have that $A \vdash B$ and $B \vdash A$ are provable in the sequent calculus. Thus, by $\Box$(re) and $\multimap R$, we have that $\vdash \Box A \multimap \Box B$ is provable in the sequent calculus. ∎

Crucially, the modal extension does not affect cut elimination.

Theorem 6. *Cut elimination holds for* MILL.

Proof. (Sketch) Cut elimination holds for Linear Logic (Girard, 1987). The proof for MILL largely adapts the proof for Linear Logic (Troelstra, 1992). Recall that the *rank* of a cut is the complexity of the cut formula. The *cutrank* of a proof is the maximum of the ranks of the cuts in the proof. The *level* of a cut is the length of the subproof ending in the cut, (Troelstra, 1992). The proof of cut elimination proceeds by induction on the cutrank of the proof. It is enough to assume that the occurrence of the cut with maximal rank is the last rule of the proof. The inductive step shows how to replace a proof ending in a cut with rank n with a proof with the same conclusion and smaller cutrank. The proof of the inductive step proceeds by induction on the level of the terminal cut, namely on the length of the proof. Note that if one of the premises of a cut is an axiom, then we can simply eliminate the cut. Given a sequent rule R, the occurrence of a formula A in the conclusion of R is principal if A has been introduced by R. As usual, there are two main cases to consider: first, the cut formula is principal in both premises of the terminal cut rule, and second, that is not the case.

First main case. We replace the cut of maximal rank with two cuts with strictly smaller rank. For example, take the case in which $\Box C$ is the cut formula and is principal in both premises (i.e. it has been introduced by $\Box$(re)):

$$\dfrac{\dfrac{B \vdash C \qquad C \vdash B}{\Box B \vdash \Box C}\,\Box(\text{re}) \qquad \dfrac{C \vdash D \qquad D \vdash C}{\Box C \vdash \Box D}\,\Box(\text{re})}{\Box B \vdash \Box D}\,\text{cut}$$

It is reduced by replacing the cut on $\Box C$ by two cuts on C with strictly smaller rank.

$$\dfrac{\dfrac{B \vdash C \qquad C \vdash D}{B \vdash D}\,\text{cut} \qquad \dfrac{D \vdash C \qquad C \vdash B}{D \vdash B}\,\text{cut}}{\Box B \vdash \Box D}\,\Box(\text{re})$$

Second main case. The cut formula is not principal in one of the premises of the cut rule. Suppose R is the rule that does not introduce the cut formula. In this case, we can apply the cut after R. By induction, the proof minus R can be turned into a cut-free proof, since the length of the subproof is smaller and the cutrank is equal or smaller. By applying again R to the cut-free proof, we obtain a proof with the same conclusion of the starting proof and less cuts.

For example,

$$\dfrac{\dfrac{B \vdash C \qquad C \vdash B}{\Box B \vdash \Box C}\,\Box(\text{re}) \qquad \dfrac{\ldots}{\Gamma, \Box C \vdash A}\,\text{R}}{\Box B, \Gamma \vdash A}\,\text{cut}$$

can be turned into:

$$\dfrac{\dfrac{\dfrac{B \vdash C \qquad C \vdash B}{\Box B \vdash \Box C}\,\Box(\text{re}) \qquad \dfrac{\ldots}{\Gamma', \Box C \vdash A'}}{\Box B, \Gamma' \vdash A'}\,\text{cut} \qquad \Gamma'' \vdash A''}{\Box B, \Gamma \vdash A}\,\text{R}$$

$\Box$

By inspecting the rules others than cut, it is easy to see that cut elimination entails the subformula property, namely if $\Gamma \vdash A$ is derivable, then there is a derivation containing subformulas of Γ and A only.

The decidability remains to be established. We can show that the proof-search for MILL is no more costly than the proof-search for propositional intuitionistic multiplicative additive linear logic (Lincoln et al., 1992).

Theorem 7. *Proof search complexity for* MILL *is in PSPACE.*

Proof. (Sketch) The proof adapts the argument in (Lincoln et al., 1992). By cut elimination, Theorem 6, for every provable sequent in MILL there is a cut-free proof with same conclusion. For every rule in MILL other than (cut), the premises have a strictly lower complexity wrt. the conclusion. Hence, for every provable sequent, there is a proof whose branches have a depth at most linear in the size of the sequent. The size of a branch is at most quadratic in the size of the conclusion. And it contains only subformulas of the conclusion sequent because of the subformula property. This means that one can non-deterministically guess such a proof, and check each branch one by one using only a polynomial space. Proof search is then in NPSPACE = PSPACE. □

We present the proof of completeness of MILL wrt. the class of modal Kripke resource frames.

Theorem 8 (Completeness of the sequent calculus)**.** *If* $\models \Gamma^* \multimap A$ *then* $\Gamma \vdash A$*.*

The proof can be summarised as follow. We build a canonical model $\mathcal{M}^c$ (Definition 3). In particular, the set M^c of states consists in the set of finite multisets of formulas, and the neutral element e^c is the empty multiset. We first need to show that it is indeed a modal Kripke resource model (Lemma 9). Second we need to show a correspondence, the "Truth Lemma", between $\vdash$ and truth in $\mathcal{M}^c$. Precisely we show that for a formula A and a multiset of formulas $\Gamma \in M^c$, it is the case that Γ satisfies A iff $\Gamma \vdash A$ is provable in the calculus (Lemma 10). Finally, to show completeness, assume that it is not the case that $\vdash \Gamma^* \multimap A$. By the Truth Lemma, it means that in the canonical model $\Gamma^* \multimap A$ is not satisfied at e^c. So $\mathcal{M}^c$ does not satisfy $\Gamma^* \multimap A$. So it is not the case that $\models \Gamma^* \multimap A$.

We construct the canonical model $\mathcal{M}^c$, then we prove that $\mathcal{M}^c$ is a modal Kripke resource model, and we prove the Truth Lemma.

In the following, $\sqcup$ is the multiset union. Also, $\mid A \mid^c = \{\Gamma \mid \Gamma \vdash A\}$.

Definition 3. *Let* $\mathcal{M}^c = (M^c, e^c, \circ^c, \geq^c, N^c, V^c)$ *such that:*

- $M^c = \{\Gamma \mid \Gamma$ *is a finite multiset of formulas*$\}$*;*
- $\Gamma \circ^c \Delta = \Gamma \sqcup \Delta$*;*
- $e^c = \emptyset$*;*
- $\Gamma \geq^c \Delta$ *iff* $\Gamma \vdash \Delta^*$*;*

- $\Gamma \in V^c(p)$ *iff* $\Gamma \vdash p$;
- $N^c(\Gamma) = \{\mid A \mid^c \mid \Gamma \vdash \Box A\}$.

Lemma 9. *$\mathcal{M}^c$ is a modal Kripke resource model.*

Proof. 1. $(M^c, e^c, \circ^c, \geq^c)$ is the "right type" of ordered monoid: (i) $(M^c, e^c, \circ^c)$ is a commutative monoid with neutral element e^c, and (ii) $\geq^c$ is a pre-order on M^c. Finally, (iii) if $\Gamma \geq^c \Delta$ and $\Gamma' \geq^c \Delta'$ then $\Gamma \circ^c \Gamma' \geq^c \Delta \circ^c \Delta'$.

For (i), commutativity (and associativity) follow from the definition of $\circ^c$ as the multiset union, and the neutrality of e^c follows from it being the empty multiset—the neutral element of the multiset union.

For (ii), $\geq^c$ is reflexive because $\{A_1, \ldots, A_n\} \vdash \{A_1, \ldots, A_n\}^*$ can be proved from the axioms (ax) $A_k \vdash A_k$, $1 \leq k \leq n$, and by applying $\otimes$R $n-1$ times. The key rule to establish that $\geq^c$ is transitive is *cut*.

For (iii), assume $\Gamma \geq^c \Delta$ and $\Gamma' \geq^c \Delta'$, that is, $\Gamma \vdash \Delta^*$ and $\Gamma' \vdash \Delta'^*$. By $\otimes$R we have $\Gamma, \Gamma' \vdash \Delta^* \otimes \Delta'^*$. By applying the definitions we end up with $\Gamma \sqcup \Gamma' \vdash (\Delta \sqcup \Delta')^*$ and the expected result follows.

2. V^c is a valuation function and satisfies heredity: if $\Gamma \in V(p)$ and $\Delta \geq^c \Gamma$ then $\Delta \in V(p)$. To see this, suppose $\Gamma \vdash p$ and $\Delta \vdash \Gamma^*$. By applying $\otimes$L enough times, we have $\Gamma^* \vdash p$. By *cut*, we obtain $\Delta \vdash p$.

3. N^c is well-defined: Suppose that $\mid A \mid^c = \mid B \mid^c$. We need to show that $\mid A \mid^c \in N^c(\Gamma)$ iff $\mid B \mid^c \in N^c(\Gamma)$.

From $\mid A \mid^c = \mid B \mid^c$, we have $\Gamma \vdash A \Rightarrow \Gamma \vdash B$. In particular, we have $A \vdash A \Rightarrow A \vdash B$. Hence, $A \vdash B$ is provable (by rule (ax)). We show symmetrically that $B \vdash A$ is provable.

From $A \vdash B$ and $B \vdash A$, we have by rule $\Box$(re) that $\Box A \vdash \Box B$ is provable, and also that $\Box B \vdash \Box A$ is provable.

Now suppose that $\Gamma \vdash \Box A$. Since $\Box A \vdash \Box B$ is provable, we obtain by *cut* that $\Gamma \vdash \Box B$ is provable. Symmetrically, suppose that $\Gamma \vdash \Box B$. Since $\Box B \vdash \Box A$ is provable, we obtain by *cut* that $\Gamma \vdash \Box A$ is provable.

Hence, we have that $\Gamma \vdash \Box A$ iff $\Gamma \vdash \Box B$. By definition of N^c, it means that $\mid A \mid^c \in N^c(\Gamma)$ iff $\mid B \mid^c \in N^c(\Gamma)$.

4. If $X \in N^c(\Gamma)$ and $\Delta \geq^c \Gamma$ then $X \in N^c(\Delta)$. To see that this is the case, the hypotheses are equivalent to $\Gamma \vdash \Box A$ for some A such that $\mid A \mid^c = X$, and $\Delta \vdash \Gamma^*$. By repeatedly applying $\otimes$L to obtain $\Gamma^* \vdash \Box A$ and by using *cut*, we infer that $\Delta \vdash \Box A$. Which is equivalent to the statement that $X \in N^c(\Delta)$. ☐

Let us then note $\models_c$ the truth relation in $\mathcal{M}^c$.

Lemma 10. $\Gamma \models_c A$ *iff* $\Gamma \vdash A$.

Proof. By induction on the complexity of A. For induction, we suppose that the lemma holds for all atomic formulas. The cases of the propositional connectives are found in (Kamide, 2003). We prove here the case $A = \Box B$.

We have the following sequence of equivalences of $\Gamma \models_c \Box B$:

iff $|| B ||^{\mathcal{M}^c} \in N^c(\Gamma)$, by definition of $\models_c$;

iff $\{\Delta \mid \Delta \models_c B\} \in N^c(\Gamma)$, by definition of $|| \, . \, ||^{\mathcal{M}^c}$;

iff $\{\Delta \mid \Delta \vdash B\} \in N^c(\Gamma)$, by Induction Hypothesis;

iff $| B |^c \in N^c(\Gamma)$, by definition of $| \, . \, |^c$;

iff $\Gamma \vdash \Box B$, by definition of N^c.

□

We could now prove that the sequent calculus is sound, and we could adapt our proof of Theorem 8 to prove that the Hilbert system H-MILL is complete. But we are already there; We have the following:

- if $\Gamma \vdash_{\textsf{H-MILL}} A$ then $e \models \Gamma^* \multimap A$ (Theorem 4);
- if $e \models \Gamma^* \multimap A$ then $\Gamma \vdash A$ (Theorem 8);
- if $\Gamma \vdash A$ then $\Gamma \vdash_{\textsf{H-MILL}} A$ (Theorem 5).

Therefore, the completeness of the Hilbert system and the soundness of the sequent calculus both follow.

Corollary 11. *We have:*

- *if* $e \models \Gamma^* \multimap A$ *then* $\Gamma \vdash_{\textsf{H-MILL}} A$*;*
- *if* $\Gamma \vdash A$ *then* $e \models \Gamma^* \multimap A$*.*

5 Adding non-commutativity

Systems that integrate a commutative Linear Logic with a non-commutative one have been studied in (Abrusci and Ruet, 1999; De Groote, 1996; Retoré, 1997). Note that the purely non-commutative version of intuitionistic Linear Logic is basically the calculus with two order sensitive implications developed by Lambek (Lambek, 1958). The basic propositional logic that we use is provided by (De Groote, 1996) and labelled PCL, partially commutative Linear Logic. The main novelty is that the structural rule of exchange no longer holds in general. The context of a sequent is now only partially commutative, and is now built by means of two constructors. Thus, we essentially use *context* as a shorthand for *partially commutative context*. Every formula (to be defined shortly after) is a context. Then, for every context Γ and Δ, we can build their *parallel* composition (Γ, Δ) (primarily commutative context) and their *serial* composition $(\Gamma; \Delta)$ (primarily non-commutative context) (Béchet et al., 1997). A context can thus be seen as finite tree with non-leaf nodes labelled with ';' or ',' and with leafs labelled by formulas. Two branches emanating from a ',' commute with each other, while the branches emanating from a ';' node do not. We write $()$ for the empty context, and assume that it acts as the identity element of the parallel and serial composition, that is: $((), \Gamma) = (\Gamma, ()) = ((); \Gamma) = (\Gamma; ()) = \Gamma$. We denote by $\Gamma[-]$ a context "with a hole", and $\Gamma[\Delta]$ denotes this very context with the "hole filled" with the context Δ.

The language of PCMILL extends the language of MILL by adding the following operators: the non-commutative tensor noted $\odot$ and the two order sensitive implications noted $\backslash$ and $/$:

$$A ::= \mathbf{1} \mid p \mid A \otimes A \mid A \,\&\, A \mid A \multimap A \mid A \odot A \mid A \backslash A \mid A/A \mid \Box A$$

where $p \in Atom$.

In order to blend together commutative and non-commutative sequence, we have to choose what is the interpretation of commutativity, namely if we view a commutative concurrent process such as $A \otimes B$ as entailing that either directions are allowed (Béchet et al., 1997; De Groote, 1996). That is, parallel composition is weaker than serial composition, so it shall hold that $A \otimes B \multimap A \odot B$. This means that if two resources can be combined with no particular order, then they can be combined sequentially. This choice is reflected by the structural rule of *entropy* (ent) below. The first two lines of Table 3 state the associativity of serial and parallel compositions, the third line states the commutativity of parallel composition and the entropy principle.

Semantics and completeness. In order to define a class of modal Kripke resource models for PCMILL, we extend the models we have considered in Section 2. We add to a modal Kripke resource model $(M, e, \circ, \geq, N, V)$ an associative, non-commutative operation $\bullet$ such that e is neutral also for $\bullet$. Thus, a Kripke resource model is now specified by $\mathcal{M} = (M, e, \circ, \bullet, \geq, N, V)$. Bifunctoriality is assumed also for $\bullet$: if $m \geq n$, and $m' \geq n'$, then $m \bullet m' \geq n \bullet n'$. Moreover, the entropy principle is captured in the models by means of the following constraint: for all x, y, $x \circ y \geq x \bullet y$. If $\mathcal{M}$ satisfies all these conditions, we call it a *partially commutative modal Kripke resource model*.

The new truth conditions are the following:

$m \models_{\mathcal{M}} A \odot B$ iff there exist m_1 and m_2 such that $m \geq m_1 \bullet m_2$ and $m_1 \models_{\mathcal{M}} A$ and $m_2 \models_{\mathcal{M}} B$.

$m \models_{\mathcal{M}} A \backslash B$ iff for all $n \in M$, if $n \models_{\mathcal{M}} A$, then $n \bullet m \models_{\mathcal{M}} B$.

$m \models_{\mathcal{M}} B/A$ iff for all $n \in M$, if $n \models_{\mathcal{M}} A$, then $m \bullet n \models_{\mathcal{M}} B$.

Note that, if $m \models A \otimes B$, then by $m_1 \circ m_2 \geq m_1 \bullet m_2$ and heredity, we have that $m \models A \odot B$.

We shall prove soundness and completeness of PCMILL. We start by discussing the partially commutative version of MILL. Soundness of PCMILL wrt.

Structural rules

$$\frac{\Gamma[\Delta_1,(\Delta_2,\Delta_3)]\vdash A}{\Gamma[(\Delta_1,\Delta_2),\Delta_3)]\vdash A}\,\text{,a1}\qquad \frac{\Gamma[(\Delta_1,\Delta_2),\Delta_3)]\vdash A}{\Gamma[\Delta_1,(\Delta_2,\Delta_3)]\vdash A}\,\text{,a2}$$

$$\frac{\Gamma[\Delta_1;(\Delta_2;\Delta_3)]\vdash A}{\Gamma[(\Delta_1;\Delta_2);\Delta_3)]\vdash A}\,\text{;a1}\qquad \frac{\Gamma[(\Delta_1;\Delta_2);\Delta_3)]\vdash A}{\Gamma[\Delta_1;(\Delta_2;\Delta_3)]\vdash A}\,\text{;a2}$$

$$\frac{\Gamma[\Delta_1,\Delta_2]\vdash A}{\Gamma[\Delta_2,\Delta_1]\vdash A}\,\text{,com}\qquad \frac{\Gamma[\Delta_1;\Delta_2]\vdash A}{\Gamma[\Delta_1,\Delta_2]\vdash A}\,\text{ent}$$

Non-commutative connectives

$$\frac{\Gamma[A;B]\vdash C}{\Gamma[A\odot B]\vdash C}\,\odot\text{L}\qquad \frac{\Gamma\vdash A\qquad \Gamma'\vdash B}{\Gamma;\Gamma'\vdash A\odot B}\,\odot\text{R}$$

$$\frac{\Gamma\vdash A\qquad \Delta[B]\vdash C}{\Delta[\Gamma;A\setminus B]\vdash C}\,\setminus\text{L}\qquad \frac{A;\Gamma\vdash B}{\Gamma\vdash A\setminus B}\,\setminus\text{R}$$

$$\frac{\Gamma\vdash A\qquad \Delta[B]\vdash C}{\Delta[B/A;\Gamma]\vdash C}\,/\text{L}\qquad \frac{\Gamma;A\vdash B}{\Gamma\vdash A/B}\,/\text{R}$$

Table 3: PCMILL: extending the sequent calculus MILL

the semantics above is just an extension of the induction for the soundness of MILL with the new rules for non-commutative connective. For completeness, we need to extend the construction of the canonical model in order to account for the non-commutative structure.

As before, a context can be associated to a unique formula by means of a recursive operation, here $.^+$. We adopt the following definition:

$$\begin{aligned}()^+ &= \mathbf{1}\\ (A)^+ &= A\\ (\Gamma,\Delta)^+ &= (\Gamma^+\otimes\Delta^+)\\ (\Gamma;\Delta)^+ &= (\Gamma^+\odot\Delta^+)\end{aligned}$$

Let $\mathcal{M}^c_\bullet = (M^c, e^c, \circ^c, \bullet^c, \geq^c, N^c, V^c)$ such that: $M^c = \{\Gamma \mid \Gamma$ is a partially commutative context$\}$; $e^c = ()$; $\Gamma\circ^c\Delta = (\Gamma,\Delta)$; $\Gamma\bullet^c\Delta = (\Gamma;\Delta)$; $\Gamma\geq^c\Delta$ iff $\Gamma\vdash\Delta^+$; $\Gamma\in V^c(p)$ iff $\Gamma\vdash p$; $N^c(\Gamma) = \{\mid A\mid^c\mid \Gamma\vdash\Box A\}$.

We will show that $\mathcal{M}^c_\bullet$ is actually a partially commutative modal Kripke resource model. It suffices to adapt and extend the proof of Lemma 9. Important bits are:

1. e^c is neutral for $\bullet^c$;

2. associativity of $\bullet^c$;

3. for all $\Gamma, \Delta \in M^c$: $\Gamma \circ^c \Delta \geq^c \Gamma \bullet^c \Delta$;

4. $\Gamma \geq^c \Delta$ and $\Gamma' \geq^c \Delta'$ then $\Gamma \bullet^c \Gamma' \geq^c \Delta \bullet^c \Delta'$;

5. If $X \in N^c(\Gamma)$ and $\Gamma \geq^c \Delta$ then $X \in N^c(\Delta)$.

We sketch the arguments here. Item 1 and item 2 follow from the definition of partially commutative contexts. We look at the case of entropy (item 3) with more attention. By repeated use of (ax), $\otimes$R, and $\odot$R, we can show $(\Gamma; \Delta) \vdash (\Gamma; \Delta)^+$. By (ent), we obtain $(\Gamma, \Delta) \vdash (\Gamma; \Delta)^+$, and apply the definition of $\geq^c$ to have $(\Gamma, \Delta) \geq^c (\Gamma; \Delta)^+$. By definition of $\circ^c$ we have $(\Gamma \circ^c \Delta) \geq^c (\Gamma; \Delta)^+$. Call In the latter inequality. Now, by (ax), we have $(\Gamma; \Delta)^+ \vdash (\Gamma; \Delta)^+$, which by definition of $\geq^c$ is equivalent to $(\Gamma; \Delta)^+ \geq^c (\Gamma; \Delta)$. By definition of $\bullet^c$, we have $(\Gamma; \Delta)^+ \geq^c (\Gamma \bullet^c \Delta)$. Together with In, we have $(\Gamma \circ^c \Delta) \geq^c (\Gamma \bullet^c \Delta)$.

To start with item 4, assume $\Gamma \geq^c \Delta$ and $\Gamma' \geq^c \Delta'$, that is by definition of $\geq^c$, $\Gamma \vdash \Delta^+$ and $\Gamma' \vdash \Delta'^+$. By $\odot$R, we have $(\Gamma; \Gamma') \vdash \Delta^+ \odot \Delta'^+$. By the definition of $.^+$, it means that $(\Gamma; \Gamma') \vdash (\Delta; \Delta')^+$. Again by definition of $\geq^c$, we have $(\Gamma; \Gamma') \geq^c (\Delta; \Delta')$. Finally by definition of $\bullet^c$ we conclude that $(\Gamma \bullet^c \Gamma') \geq^c (\Delta \bullet^c \Delta')$.

Item 5 is almost identical to the same case in the proof of Lemma 9, but we explicitly adapt it here. The hypotheses are equivalent to $\Gamma \vdash \Box A$ for some A such that $\mid A \mid^c = X$, and $\Delta \vdash \Gamma^+$. By repeatedly applying $\otimes$L and $\odot$L to obtain $\Gamma^+ \vdash \Box A$ and by using *cut*, we infer that $\Delta \vdash \Box A$. Which is equivalent to the statement that $X \in N^c(\Delta)$.

The truth lemma can be checked by routine induction. Thus we can conclude.

Theorem 12. PCMILL *is sound and complete wrt. the class of partially commutative modal Kripke resource models.*

6 Resource-sensitive "bringing-it-about"

We present the (non-normal modal) logic of agency of *bringing-it-about* (Elgesem, 1997; Governatori and Rotolo, 2005), and propose two versions of it in Linear Logic coined RSBIAT (for Resource-Sensitive "bringing-it-about") and SRSBIAT (for Resource-Sensitive "bringing-it-about" with sequences of actions). RSBIAT and SRSBIAT are respectively extensions of MILL and of PCMILL. In Section 7, we will illustrate the logic by representing a few actions of agents, functions of artefacts, and their interactions.

We specialise the minimal modality studied in the previous sections to a modality agency. In fact, for each agent a in a set $\mathcal{A}$, we define a modality E_a, and $\mathsf{E}_a A$ specifies that agent $a \in \mathcal{A}$ brings about A. As previously, to interpret them in a modal Kripke resource frame, we take one neighbourhood function N_a for each agent a that obeys Condition (1) in Definition 1. We have $m \models \mathsf{E}_a A$ iff $||A|| \in N_a(m)$.

6.1 Bringing-it-about in classical logic

The four following principles typically constitute the core of logics of agency (Belnap et al., 2001; Elgesem, 1997; Pörn, 1977):

1. If something is brought about, then this something holds.
2. It is not possible to bring about a tautology.
3. If an agent brings about two things concomitantly then the agent also brings about the conjunction of these two things.
4. If two statements are equivalent, then bringing about one is equivalent to bringing about the other.

Briefly, we explain how these principles are captured in classical logic. Item 1 is a principle of success. It corresponds to the axiom T: $\mathsf{E}_i A \to A$. Item 2 has been open to some debate, although Chellas is essentially the only antagonist. (See (Chellas, 1969) and (Chellas, 1992).) It corresponds to the axiom $\neg\mathsf{E}_i\top$ (notaut). Item 3 corresponds to the axiom: $\mathsf{E}_i A \wedge \mathsf{E}_i B \to \mathsf{E}_i(A \wedge B)$. That is, if i is doing A while also doing B, then we can deduce that i is doing $A \wedge B$. The other way round needs not be true. Item 4 confers to the concept of bringing about the quality of being a modality, effectively obeying the rule of equivalents: if $\vdash A \leftrightarrow B$ then $\vdash \mathsf{E}_i A \leftrightarrow \mathsf{E}_i B$.

6.2 Resource-sensitive BIAT

We now detail the logic of RSBIAT. We capture the four principles, adapted to the resource-sensitive framework, by means of rules in the sequent calculus, cf. Table 5

The principle of item 1 is captured by E_a(refl) that entails the linear version of T: $\mathsf{E}_a A \multimap A$. In our interpretation, it means that if an agent brings about A, then A affects the environment.

Because of the difference between the unities in Linear Logic and in classical logic, the principle of item 2 requires some attention. In classical logic all tautologies are provably equivalent to the unity $\top$. Say A is theorem ($\vdash A$), we have $\vdash A \leftrightarrow \top$. Hence, from the rule of equivalents, and the axiom $\vdash \neg\mathsf{E}_a\top$ that indicates than no agent brings about the tautological constant, one can deduce $\vdash \neg\mathsf{E}_a A$ whenever the formula A is a theorem. In Linear Logic, the unity $\mathbf{1}$ is *not* provably equivalent to all theorems. Thus, the axiom of BIAT must be changed into an inference rule ($\sim$ nec) in RSBIAT: if $\vdash A$, then $\mathsf{E}_a A \vdash \bot$. It is effectively a sort of "anti-necessitation rule". So, if a formula is a theorem, if an agent brings it about, then the contradiction is entailed. This amounts to negating $\mathsf{E}_a A$, according to intuitionistic negation, for every tautological formula A.

The principle of BIAT for combining actions (item 3 in the list) is open to two interpretations here: a multiplicative one and an additive one. The additive combination means that if there is a choice for agent a between bringing about

A and bringing about B, then agent a can bring about a choice between A and B. $\mathsf{E}_a\otimes$ means that if an agent a brings about action A and brings about action B then a brings about both actions $A \otimes B$. Moreover, in order to bring about $A \otimes B$, the sum of the resources for A and the resources for B is required.

Finally, the logics of the minimal modality already satisfy the rule of equivalents for E_a: from $A \vdash B$ and $B \vdash A$ we infer $\mathsf{E}_a A \vdash \mathsf{E}_a B$. This is inherited by RSBIAT, and it is all that is needed to capture the principle of item 4.

We enrich H-MILL, the Hilbert system for MILL, as follows and we label this system H-RSBIAT. We add the following axioms.

all axioms of H-MILL
$\mathsf{E}_a A \multimap A$
$\mathsf{E}_a A \otimes \mathsf{E}_a B \multimap \mathsf{E}_a(A \otimes B)$
$\mathsf{E}_a A \,\&\, \mathsf{E}_a B \multimap \mathsf{E}_a(A \,\&\, B)$

Table 4: Axiom schemata in H-RSBIAT

The definition of deduction in H-RSBIAT extends the definition of deduction in H-MILL (Definition 2) with the following possible rule to consider for the inductive steps.

$$\frac{\begin{array}{c}\mathcal{D}\\ \vdash_{\mathsf{H}} A\end{array}}{\vdash_{\mathsf{H}} \mathsf{E}_a A \multimap \bot}\ (\sim \text{nec})$$

On the side of the semantics, we propose the following conditions on modal Kripke resource frames $(M, e, \circ, \geq, \{N_a\}, V)$. The rule $(\sim$ nec$)$ requires:

$$\text{if } (X \in N_a(w)) \text{ and } (e \in X) \text{ then } (w \in V(\bot)) \tag{2}$$

The rule $(\mathsf{E}_a(\text{refl}))$ requires:

$$\text{if } \ X \in N_a(w) \text{ then } w \in X \tag{3}$$

The condition corresponding to the multiplicative version of action combination $(\mathsf{E}_a\otimes)$ is the following, where $X \circ Y = \{x \circ y \mid x \in X \text{ and } y \in Y\}$, and $X^{\uparrow} = \{y \mid y \geq x \text{ and } x \in X\}$:

$$\text{if } X \in N_a(x) \text{ and } Y \in N_a(y) \text{ , then } (X \circ Y)^{\uparrow} \in N_a(x \circ y) \tag{4}$$

The condition on the frames corresponding to the additive version is the following:

$$\text{if } X \in N_a(x) \text{ and } Y \in N_a(x), \text{ then } X \cap Y \in N_a(x) \tag{5}$$

Next, we introduce a sequent calculus for RSBIAT. The rules of RSBIAT for E& show that in order to bring about the choice between $A \,\&\, B$ is enough to use the resources for one of the two. On the contrary, in order to bring about $A \otimes B$, the sum of the resources for A and the resources for B is required.

We can prove that H-RSBIAT and the sequent calculus for RSBIAT are equivalent.

$$\frac{\vdash A}{\mathsf{E}_a A \vdash \bot}\ \sim\text{nec} \qquad \frac{\Gamma \vdash \mathsf{E}_a A \qquad \Gamma \vdash \mathsf{E}_a B}{\Gamma \vdash \mathsf{E}_a(A \,\&\, B)}\ \mathsf{E}_a\&$$

$$\frac{\Gamma, A \vdash B}{\Gamma, \mathsf{E}_a A \vdash B}\ \mathsf{E}_a(\text{refl}) \qquad \frac{\Gamma \vdash \mathsf{E}_a A \qquad \Delta \vdash \mathsf{E}_a B}{\Gamma, \Delta \vdash \mathsf{E}_a(A \otimes B)}\ \mathsf{E}_a\otimes$$

Table 5: RSBIAT (extends MILL)

Proposition 13. *It holds that* $\Gamma \vdash_{\mathsf{H\text{-}RSBIAT}} A$ *iff* $\Gamma \vdash A$ *is derivable in* RSBIAT.

Proof. (Sketch) The proof is again an induction on the length of derivations. For example, in one direction, axiom $\mathsf{E}_a A \otimes \mathsf{E}_a B \multimap \mathsf{E}_a(A \otimes B)$ is derivable in the sequent calculus by simply applying $\otimes L$ and $\multimap R$ to $\mathsf{E}_a A, \mathsf{E}_a B \vdash \mathsf{E}_a(A \otimes B)$ which has been obtained from axioms by means of $\mathsf{E}_a\otimes$. □

We can now prove soundness and completeness of RSBIAT.

Theorem 14. RSBIAT *is sound and complete wrt. the class of modal Kripke frames that satisfy (2), (3), (4), and (5).*

Proof. (Sketch) We just show two correspondences.

Condition (2) and rule (∼nec). (∼nec) is sound. Assume that for every model, $e \models A$. We need to show that $e \models \mathsf{E}_a A \multimap \bot$. That is, for every x, if $x \models \mathsf{E}_a A$, then x models $\bot$. If $x \models \mathsf{E}_a A$, then by definition, $||A|| \in N_a(x)$. Since A is a theorem, $e \in ||A||$, thus by Condition 2, $x \in V(\bot)$, so $x \models \bot$. For completeness, it suffices to adapt our canonical model construction. Build the canonical model for RSBIAT as in Def. 3 (we have now more valid sequents). Now suppose (1) $X \in N^c_a(\Gamma)$, and (2) $e^c \in X$. By definition of N^c_a and of $|\ .\ |^c$, there is A, s.t. $|\ A\ |^c = X$, (1) $\Gamma \vdash \mathsf{E}_a A$ and (2) $\vdash A$. From (2), and (∼nec): $\mathsf{E}_a A \vdash \bot$. From (1), and previous, we obtain $\Gamma \vdash \bot$ using (cut). By definition of V^c, $\Gamma \in V^c(\bot)$.

Condition (4) and rule ($\mathsf{E}_a\otimes$). ($\mathsf{E}_a\otimes$) is sound. Assume $e \models \Gamma^* \multimap \mathsf{E}_a A$ and $e \models \Delta^* \multimap \mathsf{E}_a B$. Then, for all x that make Γ true, $||A|| \in N_a(x)$ and for all y that make Δ true, $||B|| \in N_a(y)$. By (4), $||A|| \circ ||B|| \in N_a(x \circ y)$, so for any $x \circ y$ that make $(\Gamma, \Delta)^*$ true, $x \circ y \models \mathsf{E}_a(A \otimes B)$. For completeness, suppose $X \in N^c_a(\Gamma)$ and $Y \in N^c_a(\Delta)$. By definition of N^c_a and of $|\ .\ |^c$, there is A and B, with $|\ A\ |^c = X$, and $|\ B\ |^c = Y$, s.t. $\Gamma \vdash \mathsf{E}_a A$, and $\Delta \vdash \mathsf{E}_a B$. By ($\mathsf{E}_a\otimes$), we obtain $\Gamma, \Delta \vdash \mathsf{E}_a(A \otimes B)$ and thus $|\ A \otimes B\ |^c \in N^c_a(\Gamma \sqcup \Delta)$ by definition of $|\ .\ |^c$. The definition of $\circ^c$ gives us $|\ A \otimes B\ |^c \in N^c_a(\Gamma \circ^c \Delta)$. By the Truth Lemma, we have that $||A \otimes B||^{\mathcal{M}^c} \in N^c_a(\Gamma \circ^c \Delta)$. Thus $(X \circ^c Y)^\uparrow \in N^c_a(\Gamma \circ^c \Delta)$.

□

Therefore, also our extensions of Hilbert systems and sequent calculus are sound and complete wrt. the modal Kripke resource frames restricted to the relevant conditions given in this section.

Corollary 15. H-RSBIAT *is sound and complete wrt. the class of modal Kripke frames that satisfy (2), (3), and (4).*

Moreover, RSBIAT enjoys cut elimination.

Theorem 16. *Cut elimination holds for* RSBIAT*.*

Proof. (Sketch) We extend the proof of Theorem (6) by presenting a number of new cases for the cut formula being principal in both premises of the cut rule. The other cases can be treated similarly. The cut formula has been introduced by E_a(re) and ∼nec

$$\dfrac{\dfrac{A \vdash B \quad B \vdash A}{\mathsf{E}_a A \vdash \mathsf{E}_a B}\,\mathsf{E}_a(\text{re}) \qquad \dfrac{\vdash B}{\mathsf{E}_a B \vdash \bot}\,{\sim}\text{nec}}{\mathsf{E}_a A \vdash \bot}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash B \quad B \vdash A}{\vdash A}\,\text{cut}}{\mathsf{E}_a A \vdash \bot}\,{\sim}\text{nec}$$

The cut formula has been introduced by $\mathsf{E}_a\otimes$ and E_a(refl).

$$\dfrac{\dfrac{\dfrac{\Gamma' \vdash A \quad \ldots}{\Gamma \vdash \mathsf{E}_a A} \quad \dfrac{\Delta' \vdash B \quad \ldots}{\Delta \vdash \mathsf{E}_a B}}{\Gamma, \Delta \vdash \mathsf{E}_a(A \otimes B)} \qquad \dfrac{\dfrac{\Sigma', A \vdash C' \quad \Sigma'', B \vdash C''}{\Sigma, A \otimes B \vdash C}}{\Sigma, \mathsf{E}_a(A \otimes B) \vdash C}}{\Gamma, \Delta, \Sigma \vdash C}\,\text{cut}$$

It can be reduced by pushing the cut upwards.

$$\dfrac{\dfrac{\dfrac{\Gamma' \vdash A \quad \Sigma', A \vdash C'}{\Gamma', \Sigma' \vdash C'}\,\text{cut} \qquad \dfrac{\Delta' \vdash B \quad \Sigma'', B \vdash C''}{\Delta', \Sigma'' \vdash C''}\,\text{cut}}{\ldots}\,\ldots}{\Gamma, \Delta, \Sigma \vdash C}\,\ldots$$

□

Once again, it is easy to see that cut elimination entails the subformula property for RSBIAT. Using the same arguments as for Theorem 7, it is clear that we can decide polynomial space whether a sequent is valid in RSBIAT.

Theorem 17. *Proof search complexity for* RSBIAT *is in PSPACE.*

6.3 RSBIAT with sequences of actions

So far, we have discussed how to control weakening and contraction in order to provide a resource sensitive account of agency. There is one important structural rule that we did not discuss, namely the exchange rule. In this section, we extend RSBIAT by introducing the *non-commutative* multiplicative conjunction $\odot$, and its two associated order-sensitive implications.

The significance of this move goes beyond the technical aspect, which is rather straightforward at that point. Indeed, a recurring point of contention against the logics of bringing-it-about is the absence of a basic notion of time.

Non-commutative composition of formulas will provide to us an immediate and natural way of talking, if not about time proper, at least about sequences of actions. Reading $A \odot B$ as "first A then B", we will also read $(\mathsf{E}_a A) \odot (\mathsf{E}_b B)$ as "first a brings about A then b brings about B".

We obtain SRSBIAT by adding the non-commutative versions of $\mathsf{E}_a\otimes$ and rephrasing the commutative rules by means of the context notation. The rule $\mathsf{E}_a(\text{refl})$ can now operate both in commutative and non-commutative contexts.

$$\frac{\Gamma[A] \vdash B}{\Gamma[\mathsf{E}_a A] \vdash B}\ \mathsf{E}_a(\text{refl}) \qquad \frac{\Gamma \vdash \mathsf{E}_a A \qquad \Delta \vdash \mathsf{E}_a B}{\Gamma ; \Delta \vdash \mathsf{E}_a (A \odot B)}\ \mathsf{E}_a\odot$$

Table 6: Resource "bringing-it-about" with sequences of actions (extends PCL and H-RSBIAT)

Note that the presentation of $\sim$nec, $\mathsf{E}_a\otimes$, and $\mathsf{E}_a\&$ is not affected by the generalisation to partially commutative. $\mathsf{E}_a(\text{refl})$ states that we can introduce the modality also in non-commutative contexts. $\mathsf{E}_a\odot$ adapts the principle of composition of actions (cf. item 3, p. 17) to sequences; it states that we can compose two ordered actions into one sequence of actions.

In order to offer a semantics to our extension of RSBIAT with sequences of actions, we need to add a condition on the neighbourhood functions that deals with the non-commutative operator. Specifically, we need the following condition:

$$\text{if } X \in N_a(x) \text{ and } Y \in N_a(y)\text{ , then } (X \bullet Y)^{\uparrow} \in N_a(x \bullet y) \tag{6}$$

We obtain naturally the semantic determination.

Theorem 18. SRSBIAT *is sound and complete wrt. the class of partially commutative modal Kripke resource models that satisfy (2), (3), (4), (5), and (6).*

7 Application: manipulation of artefacts

7.1 Artefacts

Our application lies in the reasoning about artefact's function and tool use. By endorsing what we may call an *agential* stance, we view artefacts as special kind of agents. They are characterised by the fact that they are designed by some other agent in order to achieve a purpose in a particular environment. An important aspect of the modelling of artefacts is their interaction with the environment and with the agents that *use* the artefact to achieve a specific goal (Borgo and Vieu, 2009; Garbacz, 2004; Houkes and Vermaas, 2010; Kroes, 2012). Briefly, we can view an artefact as an object that in presence of a number of preconditions $c_1, \dots, c_n$ produces the outcome o. In this work, we want to represent the function of artefacts by means of logical formulas and to view the correct behaviour of an artefact by means of a form of reasoning. When reasoning about artefacts and their outcomes, we need to be careful in making all the conditions of use of the artefact explicit, otherwise we end up facing the following unintuitive cases. Imagine we represent the behaviour of a screwdriver as a formula of classical logic that states that if there is a screw S, then we can tighten it T. We simply describe the behaviour of the artefact as a material implication $S \to T$. In classical logic, we can infer that by means of a single screwdriver we can tighten two screws: $S, S, S \to T \vdash T \wedge T$. Worse, we do not even need to have two screws to begin with: $S, S \to T \vdash T \wedge T$. Thus, without specifying all the relevant constraints on the environment (e.g., that a screwdriver can handle one screw at the time) we end up with unintuitive results. Another possible drawback of classical logic is that it is commutative, the order of formulas does not matter. For example, if we describe the process of hammering a nail by means of the implication if I place a nail N and I provide the right force F, then I can drive a nail in (D), that is $N \wedge F \to D$, that would entail also that one can put a force before placing the nail.

Moreover, we need to specify the relationship between the artefact and the agents: for example, there are artefacts that can be used by one agent at the time. Since a crucial point in modelling artefacts is their interaction with the environment, either we carefully list all the relevant conditions, or we need to change the logical framework that we use to represent the artefact's behaviour. In this paper, we propose to pursue this second strategy. Our motivation is that, instead of specifying for each artefact the precondition of its application (e.g., that there is only one screw that a screw driver is supposed to operate on), the logical language that encodes the behaviour of the artefact already takes care of preventing unintuitive outcomes. Thus, the formulas of Linear Logic shall represent actions of agents and functions of artefacts, and the non-normal modality shall specify which agent or artefact brings about which process.

7.2 Functions

The concept of a *function* of an artefact aims to capture the description of the behaviour of an artefact in an environment with respect to its goals: Artefacts are not living things but have a purpose, attributed by a designer or a user (Borgo and Vieu, 2009; Kroes, 2012). We model a function of an artefact by means of a formula A in RSBIAT or SRSBIAT. If A is a function of an artefact t, then one can represent t's behaviour as $\mathsf{E}_t A$ (t brings about A) in a conceptually consistent manner, namely an artefact brings about its function A.

With Linear Logic, we are equipped with a formalism to represent and reason about processes and resources. In classical and intuitionistic logic, if one has A and A implies B, then one has B, but A still holds. This is fine for mathematical reasoning but often fails to be acceptable in the real world where implication is *causal*. Girard remarks that "[a] causal implication cannot be iterated since the conditions are modified after its use; this process of modification of the premises (conditions) is known in physics as *reaction*." (Girard, 1989, p. 72)

That is, Linear Logic allows for modelling how the function of an artefact can be actually realised in a certain environment. At an abstract level, an artefact can be seen as an agent t. It takes resource-sensitive actions by reacting to the environment. When t is an artefact, and $\Gamma \vdash \mathsf{E}_i A$ is provable, then also $\Gamma \vdash A$ is provable, thus the occurrence of A in the proof represents the execution of a function of t in the environment Γ. For any artefact t with function A, $\mathsf{E}_t A$, we say that t accomplishes a certain goal O in the environment Γ if and only if the sequent $\Gamma[\mathsf{E}_t A] \vdash O$ is provable. The context Γ describes a number of preconditions that specify the environment resources as well as the actions of the agents that are interacting with the artefact.

This view simply generalises to a number of artefacts as follows:

$$\Gamma[\mathsf{E}_1 A_1, \ldots, \mathsf{E}_m A_m] \vdash O$$

Again, if the above sequent is provable, then the combination of artefacts $\mathsf{E}_1 A_1$, ..., $\mathsf{E}_m A_m$ can achieve the goal O in Γ by executing their functions $A_1, \ldots, A_m$.

Defining the function of an artefact as a formula demands some care because in this way functions do not have a unique formulation. The functions $(A \otimes B) \multimap C$, and $A \multimap (B \multimap C)$ are provably equivalent. However, the rule E_a(re) ensures that bringing about a function is provably equivalent to bringing about any of its equivalent forms.
By means of sequent calculus provability, we can view the problem of using artefacts in an environment to achieve a goal as a decision problem that is related to the AI problem of planning (Kanovich and Vauzeilles, 2001). Note that the complexity of deciding whether a goal is achievable depends only on the fragment of the logic that we use to model the formulas in the sequent. In the next paragraphs, we shall instantiate the descriptive features of our calculus by means of a number of toy example.

7.3 Simple examples of functions

Take a very simple example. We can represent the function of a screwdriver s as an implication that states that if there is a screw (formula S) and some agent brings about the right rotational force (F), then the screw gets tighten (T). The formula corresponding to the function of the screwdriver is $S \otimes F \multimap T$. The formula that captures the screwdriver as an agent of the system is $\mathsf{E}_s(S \otimes F \multimap T)$.

Suppose the environment provides S and an agent i is providing the right force $\mathsf{E}_i F$. We can show by means of the following proof in RSBIAT that the goal T can be achieved.

$$\dfrac{\dfrac{\dfrac{S \vdash S \qquad \dfrac{F \vdash F}{\mathsf{E}_i F \vdash F}\ \mathsf{E}_i(\text{refl})}{S, \mathsf{E}_i F \vdash S \otimes F}\ \otimes\text{R} \qquad T \vdash T}{S, \mathsf{E}_i F, S \otimes F \multimap T \vdash T}\ \multimap\text{L}}{S, \mathsf{E}_i F, \mathsf{E}_s(S \otimes F \multimap T) \vdash T}\ \mathsf{E}_s(\text{refl})$$

Our calculus is resource sensitive, thus, as expected, we cannot infer for example that two agents can use the same screwdriver at the same time to tighten two screws:

$$S, S, \mathsf{E}_i F, \mathsf{E}_j F, \mathsf{E}_s(S \otimes F \multimap T) \nvdash T \otimes T$$

Often, to be effective for some goal B, an artefact's function transforming a resource A into a resource B should not be realised before the resource A is available.

In the case of our example, the description of a screwdriver should exclude that the screw can be tighten *before* a loose screw and a rotational force, in this order, are provided. Thus, we may reconsider our screwdriver as a "non-commutative" screwdriver $s\bullet$ and write its function as $S \odot F \setminus T$. The screwdriver is now defined as $\mathsf{E}_{s\bullet}(S \odot F \setminus T)$.

$$\dfrac{\dfrac{\dfrac{S \vdash S \qquad \dfrac{F \vdash F}{\mathsf{E}_i F \vdash F}\,\mathsf{E}_i(\text{refl})}{S; \mathsf{E}_i F \vdash S \odot F}\,\odot\text{R} \qquad T \vdash T}{(S; \mathsf{E}_i F); S \odot F \setminus T \vdash T}\,\setminus\text{L}}{S; \mathsf{E}_i F; \mathsf{E}_{s\bullet}(S \odot F \setminus T) \vdash T}\,\mathsf{E}_{s\bullet}(\text{refl})$$

The meaning of entropy (ent) is the following. By means of (ent), we can infer a fully commutative context:

$$S, \mathsf{E}_i F, \mathsf{E}_{s\bullet}(S \odot F \setminus T) \vdash T$$

That means that it is the description of the function of the artefact that takes care of of specifying how the resources have to be ordered. Of course, our screwdriver formula correctly excludes: $S; \mathsf{E}_{s\bullet}(S \odot F \setminus T); \mathsf{E}_i F \nvdash T$, $\mathsf{E}_i F; \mathsf{E}_{s\bullet}(S \odot F \setminus T); S \nvdash T$, and $\mathsf{E}_i F; S; \mathsf{E}_{s\bullet}(S \odot F \setminus T) \nvdash T$.

The interaction of commutative and non-commutative operators is exemplified as follows. Suppose there are two screws S, S, two screwdrivers s and s' and two agents a, b. The goal of tightening two screws can be achieved by using the screwdrivers in whatever order, as the following proof shows.

$$\dfrac{\dfrac{\dfrac{\dfrac{S \vdash S \quad \dfrac{F \vdash F}{\mathsf{E}_a F \vdash F}\,\mathsf{E}_i(\text{refl})}{S; \mathsf{E}_a F \vdash S \odot F}\,\odot\text{R} \quad T \vdash T}{(S; \mathsf{E}_a F); S \odot F \setminus T \vdash T}\,\setminus\text{L}}{S; \mathsf{E}_a F; \mathsf{E}_s(S \odot F \setminus T) \vdash T}\,\mathsf{E}_s(\text{refl}) \qquad \dfrac{\dfrac{\dfrac{S \vdash S \quad \dfrac{F \vdash F}{\mathsf{E}_b F \vdash F}\,\mathsf{E}_b(\text{refl})}{S; \mathsf{E}_b F \vdash S \odot F}\,\odot\text{R} \quad T \vdash T}{(S; \mathsf{E}_b F); S \odot F \setminus T \vdash T}\,\setminus\text{L}}{S; \mathsf{E}_b F; \mathsf{E}_{s'}(S \odot F \setminus T) \vdash T}\,\mathsf{E}_{s'}(\text{refl})}{[S; \mathsf{E}_a F; \mathsf{E}_s(S \odot F \setminus T)], [S; \mathsf{E}_b F; \mathsf{E}_{s'}(S \odot F \setminus T)] \vdash T \otimes T}\,\otimes\text{R}$$

7.4 Functions composition

By extending the previous example, we can demonstrate how the output of some artefact's function can naturally be fed into another function so as to construct a new complex artefact.

An *electric screwdriver* has two components. Firstly, the *power-pistol* creates some rotational force F when the button is pushed (P): $P \setminus F$. Secondly, what is typically called the screwdriver *bit* is for all intents and purposes effectively a screwdriver as specified before: it tightens a loose screw when a rotational force is applied. We define the electric screwdriver by means of $\mathsf{E}_e((P \setminus F) \odot (S \odot F \setminus T))$

Now suppose the environment provides a loose screw S and an agent i is pushing the button of the power pistol: $\mathsf{E}_i P$. We can show again that the goal T of having a tighten screw can be achieved, by using the electric screwdriver.

$$
\dfrac{\dfrac{\dfrac{\dfrac{S \vdash S \qquad \dfrac{\dfrac{P \vdash P}{\mathsf{E}_i P \vdash P}\,\mathsf{E}_i(\text{refl}) \qquad F \vdash F}{\mathsf{E}_i P; P \setminus F \vdash F}\,\setminus\text{L}}{S; \mathsf{E}_i P; P \setminus F \vdash S \odot F}\,\odot\text{R} \qquad T \vdash T}{S; \mathsf{E}_i P; P \setminus F, S \odot F \setminus T \vdash T}\,\setminus\text{L}}{S; \mathsf{E}_i P; (P \setminus F) \odot (S \odot F \setminus T) \vdash T}\,\odot\text{L}}{S; \mathsf{E}_i P; \mathsf{E}_e((P \setminus F) \odot (S \odot F \setminus T)) \vdash T}\,\mathsf{E}_e(\text{refl})
$$

7.5 Complex interactions between agents and artefacts

The function of an artefact may require to specify how agents use it. A number of aspects of the interaction between agents' actions and artefacts can be captured by means of the rules $\mathsf{E}_i\otimes$, $\mathsf{E}_i\odot$, and $\mathsf{E}_i\&$. Recall that $\mathcal{A}$ is a set of agents. We write $\mathop{\&}_{x\in\mathcal{A}} \mathsf{E}_x A$ as a short hand for $\mathsf{E}_{i_1} A \,\&\, \cdots \,\&\, \mathsf{E}_{i_m} A$. The latter formula means that any agent can perform A, so for example $\mathsf{E}_{i_1} A \,\&\, \cdots \,\&\, \mathsf{E}_{i_m} A \vdash \mathsf{E}_{i_j} A$.

An artefact that is defined by $\mathsf{E}_t(\mathop{\&}_{x\in\mathcal{A}}(\mathsf{E}_x(A \otimes B) \multimap O))$ requires the *same* agent x to perform both actions A and B in order to get O. For example, a one person rowboat that requires a single agent to operate on both oars (R_1) and (R_2), in whatever order, so to produce movement (M). This is can be modelled by means of our $\mathsf{E}_i\otimes$ rule.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{E}_i R_1 \vdash \mathsf{E}_i R_1 \qquad \mathsf{E}_i R_2 \vdash \mathsf{E}_i R_2}{\mathsf{E}_i R_1, \mathsf{E}_i R_2 \vdash \mathsf{E}_i(R_1 \otimes R_2)}\,\mathsf{E}_i\otimes \qquad M \vdash M}{\mathsf{E}_i R_1, \mathsf{E}_i R_2, \mathsf{E}_i((R_1 \otimes R_2) \multimap M) \vdash M}\,\multimap\text{L}}{\mathsf{E}_i R_1, \mathsf{E}_i R_2, \mathop{\&}_{x\in\mathcal{A}} \mathsf{E}_i((R_1 \otimes R_2) \multimap M) \vdash M}\,\&\text{L enough times}}{\mathsf{E}_i R_1, \mathsf{E}_i R_2, \mathsf{E}_t(\mathop{\&}_{x\in\mathcal{A}} \mathsf{E}_i((R_1 \otimes R_2) \multimap M)) \vdash M}\,\mathsf{E}_t(\text{refl})
$$

On the other hand, by specifying the function by $\mathsf{E}_t(\mathop{\&}_{x,y\in\mathcal{A},x\neq y}(\mathsf{E}_x A \otimes \mathsf{E}_y B) \multimap O)$, we are forcing the agents who operate tool t to be different (e.g., a crosscut saw). If an artefact's function does not determine whether the actions must be performed by the same agent, we can write $\mathsf{E}_t(\mathop{\&}_{x,y\in\mathcal{A}}(\mathsf{E}_x A \otimes \mathsf{E}_y B) \multimap O)$.

In the non-commutative case, $\mathsf{E}_t(\mathop{\&}_{x\in\mathcal{A}} \mathsf{E}_x(A \odot B) \multimap O)$ forces the same agent to perform first A and then B, whereas $\mathsf{E}_t(\mathop{\&}_{x,y\in\mathcal{A},x\neq y}(\mathsf{E}_x A \odot \mathsf{E}_y B) \multimap O)$ forces the agents to be different. For example, the function of a hanging ladder is described as follows: firstly, an agent holds the laden (Ho), then another agent climbs up (Cl) and reaches a certain position R: $\mathsf{E}_t(\mathsf{E}_a Ho \odot \mathsf{E}_b Cl \setminus \mathsf{E}_b R)$.

$$\dfrac{\dfrac{\dfrac{\mathsf{E}_a Ho \vdash \mathsf{E}_a Ho \qquad \mathsf{E}_b Cl \vdash \mathsf{E}_b Cl}{\mathsf{E}_a Ho; \mathsf{E}_b Cl \vdash \mathsf{E}_a Ho \odot \mathsf{E}_b Cl}\odot\text{R} \qquad \mathsf{E}_b R \vdash \mathsf{E}_b R}{\mathsf{E}_a Ho; \mathsf{E}_b Cl; \mathsf{E}_a Ho \odot \mathsf{E}_b Cl \setminus \mathsf{E}_b R \vdash \mathsf{E}_b R}\setminus\text{L}}{\mathsf{E}_a Ho; \mathsf{E}_b Cl; \mathsf{E}_t(\mathsf{E}_a Ho \odot \mathsf{E}_b Cl \setminus \mathsf{E}_b R) \vdash \mathsf{E}_b R}\,\mathsf{E}_t(\text{refl})$$

By means of $\mathsf{E}_i\&$, we can describe a function that requires an agent's choice. For example, a monkey wrench can tighten two sizes of nuts/bolts (N_1, N_2) provided that an agent chooses the right measure (M_1, M_2): $\mathsf{E}_t((\mathsf{E}_i(M_1 \& M_2) \multimap \mathsf{E}_i N_1) \& (\mathsf{E}_i(M_1 \& N_2) \multimap \mathsf{E}_i N_2))$. The following proof shows that if a single agent can choose the right measure for the nut (e.g., M_1), then the same agent can tighten the right type of nut (e.g., $\mathsf{E}_i N_1$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{E}_i M_1 \vdash \mathsf{E}_i M_1}{\mathsf{E}_i M_1 \& \mathsf{E}_i M_2 \vdash \mathsf{E}_i M_1}\&\text{L} \qquad \dfrac{\mathsf{E}_i M_2 \vdash \mathsf{E}_i M_2}{\mathsf{E}_i M_1 \& \mathsf{E}_i M_2 \vdash \mathsf{E}_i M_2}\&\text{R}}{\mathsf{E}_i M_1 \& \mathsf{E}_i M_2 \vdash \mathsf{E}_i(M_1 \& M_2)}\mathsf{E}_i\& \qquad \mathsf{E}_i N_1 \vdash \mathsf{E}_i N_1}{\mathsf{E}_i M_1 \& \mathsf{E}_i M_2, \mathsf{E}_i(M_1 \& M_2) \multimap \mathsf{E}_i N_1 \vdash \mathsf{E}_i N_1}\multimap\text{L}}{\mathsf{E}_i M_1 \& \mathsf{E}_i M_2, (\mathsf{E}_i(M_1 \& M_2) \multimap \mathsf{E}_i N_1) \& (\mathsf{E}_i(M_1 \& M_2) \multimap \mathsf{E}_i N_2) \vdash \mathsf{E}_i N_1}\&\text{L}}{\mathsf{E}_i M_1 \& \mathsf{E}_i M_2, \mathsf{E}_t((\mathsf{E}_i(M_1 \& M_2) \multimap \mathsf{E}_i N_1) \& (\mathsf{E}_i(M_1 \& M_2) \multimap \mathsf{E}_i N_2)) \vdash \mathsf{E}_i N_1}\mathsf{E}_t(\text{refl})$$

In a similar way, we can represent functions of artefacts that require any number of actions and of agents to achieve a goal (of course, if we want to express that any subsets of $\mathcal{A}$ can operate the tool, then we need an exponentially long formula).

7.6 Function warranty and reuse of artefacts

RSBIAT is resource-sensitive as the non-provable sequent in our screwdriver example illustrated (Sec. 7.3)

$$S, S, \mathsf{E}_i F, \mathsf{E}_j F, \mathsf{E}_s(S \otimes F \multimap T) \nvdash T \otimes T$$

The screwdriver *cannot be reused*, despite the fact that an additional screw is available and an appropriate force is brought about. This is perfectly fine as long as our interpretation of resource consumption is *concurrent*: all resources are consumed at once. And indeed, one cannot tighten two screws at once with only one screwdriver.

Abandoning a concurrent interpretation of resource consumption, we may specialise the modality E_a when a is an artefactual agent in such a way that the function of an artefact can be used at will. After all, using a screwdriver once does not destroy the screwdriver. Its function is still present after. We are after a property of *contraction* for our operator E_s.

$$\dfrac{\Gamma, \mathsf{E}_s A, \mathsf{E}_s A \vdash B}{\Gamma, \mathsf{E}_s A \vdash B}\,\text{c}(\mathsf{E}_s)$$

Now, if we adopt the rule, c(E_s) we can easily see that indeed

$$S, S, \mathsf{E}_i F, \mathsf{E}_j F, \mathsf{E}_s(S \otimes F \multimap T) \vdash T \otimes T$$

is provable.

There are several issues with this solution to 'reuse' as a duplication of assumptions. Some technical, some conceptual. The main technical issue is that we lose a lot of control on the proof search, as contraction is the main source of non-termination (of bottom-up proof search). Another technical (or theoretical) issue is that trying to give a natural condition on our frames that would be canonical for contraction is out of question. The conceptual issue is the same as the one posed by Girard in creating Linear Logic: duplication of assumptions should not be automatic. Similarly, *ad lib* reuse of an artefact does not reflect a commonsensical experience. In general, although they don't consume after the first use, tools will nonetheless eventually become so worn out that they will not realise their original function.

We can capitalise on the 'additive' feature of Linear Logic language: employing the 'with' operator &, we can specify a sort of warranty of artefact functions. Denote $A^n = A \odot \cdots \odot A$, for n times. We present the treatment by focusing on our example of screwdriver. A sequentially reusable screwdriver is defined as follows:

$$(S \odot F \setminus T)^{\leq n} = (S \odot F \setminus T) \mathbin{\&} (S^2 \odot F^2 \setminus T^2) \mathbin{\&} \cdots \mathbin{\&} (S^n \odot F^n \setminus T^n)$$

For example, with three screws, and three agents (a, b, and c) providing the appropriate force, then using a decently robust screwdriver, one can obtain three tighten screws. We have:

$$S, S, S, \mathsf{E}_a F, \mathsf{E}_b F, \mathsf{E}_c F, \mathsf{E}_{s\bullet}((S \odot F \setminus T)^{\leq 10000}) \vdash T \odot T \odot T$$

where $s\bullet$ is our "non-commutative" screwdriver, now with a ten thousand-use warranty.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{S \vdash S \quad S \vdash S}{S; S \vdash S \odot S} \quad S \vdash S}{S; S; S \vdash S \odot S \odot S} \quad \dfrac{\dfrac{\mathsf{E}_a F \vdash F \quad \mathsf{E}_b F \vdash F}{\mathsf{E}_a F; \mathsf{E}_b F \vdash \mathsf{E}_a F \odot \mathsf{E}_b F} \quad \mathsf{E}_c F \vdash F}{\mathsf{E}_a F; \mathsf{E}_b F; \mathsf{E}_c F \vdash F \odot F \odot F}}{S; S; S; \mathsf{E}_a F; \mathsf{E}_b F; \mathsf{E}_c F \vdash S \odot S \odot S \odot F \odot F \odot F} \quad T^3 \vdash T^3}{S; S; S; \mathsf{E}_a F; \mathsf{E}_b F; \mathsf{E}_c F; S^3 \odot F^3 \setminus T^3 \vdash T^3}}{S; S; S; \mathsf{E}_a F; \mathsf{E}_b F; \mathsf{E}_c F; (S \odot F \setminus T)^{\leq 10000} \vdash T^3}\,\&\text{L}}{S; S; S; \mathsf{E}_a F; \mathsf{E}_b F; \mathsf{E}_c F; \mathsf{E}_s((S \odot F \setminus T)^{\leq 10000}) \vdash T^3}\,\mathsf{E}_s(\text{re})$$

Note that, the goal $T \otimes T \otimes T$ is not provable, and this reflects our view of reusability as a sequential operation.

By pushing this analogy of warranty of artefact functions further on, we can model a "refurbishing" function that augments the warranty of the function A of a tool t. For instance, consider the refurbishing function which at the cost of consuming a resource R, transforms a worn out (but not too much worn out!)

screwdriver t into a screwdriver t with a function with extended warranty. It can be written as:

$$R \otimes \mathsf{E}_s((S \odot F \setminus T)^{\leq 50}) \multimap \mathsf{E}_s((S \odot F \setminus T)^{\leq 7000})$$

and is for example a function that a bench grinder would have.

8 Conclusion

The semantics of all the languages studied in this paper are adequate extensions of Urquhart's Kripke resource models for intuitionistic substructural logic. We first enriched the Kripke resource models with a neighbourhood function to give a meaning to a minimal (non-normal) modality. We obtained what we simply coin modal Kripke resource models. We thus defined and studied a minimal non-normal modal logic. The non-normal minimal modality $\Box$ is defined in the usual way where $\Box A$ holds at a point of evaluation iff the extension of A is in the neighbourhood of the point of evaluation. With Condition 1 on the models we enforced a sort of heredity (or monotonicity) on the neighbourhood function, without which the logic would not even validate modus ponens.

We introduced a Hilbert system and a sequent calculus. We showed that both are sound and complete with respect to the class of modal Kripke resource models. Moreover, we showed that the sequent calculus allows cut elimination. We used this fact to establish that proof search can be done in PSPACE. The soundness and completeness of the sequent calculus, and the cut-elimination show that the rule $\Box$(re)

$$\frac{A \vdash B \qquad B \vdash A}{\Box A \vdash \Box B}$$

which is atypical in a sequent calculus, yields the expected results and presents no logical issue. Moreover, the new semantics and calculi are general enough to extend the framework, as we did, to account for partially commutative Linear Logic.

The significance of non-normal modal logics and their semantics in modern developments in logics of agents has been emphasised before in the literature. In classical logic, logics of agency in particular, have been widely studied and used in practical philosophy and in multi-agent systems. Moving from classical logic to resource-sensitive logics allows to lift the study of agents bringing about states of affairs to the study of agents bringing about resources, or of artefacts bringing about resource transformations.

We thus instantiated the minimal modal logics (commutative / and partially commutative) with a resource-sensitive version of the logics of bringing-it-about. Again, sound and complete Hilbert system and sequent calculus are provided. Proof-search in the commutative version is shown to be PSPACE-easy. We finally presented a number of applications of the resulting resource-conscious logics of agency to reason about the resource-sensitive manipulations of technical artefacts.

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